SPHERICAL
TRIGONOMETRY
TODHUNTER
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SPHERICAL TRIGONOMETRY

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FOR THE USE OF COLLEGES AND SCHOOLS

BY THE LATE

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PREFACE.

The present work is constructed on the same plan as my treatise on Plane Trigonometry, to which it is intended as a sequel; it contains all the propositions usually included under the head of Spherical Trigonometry, together with a large collection of examples for exercise. In the course of the work reference is made to preceding writers from whom assistance has been obtained; besides these writers I have consulted the treatises on Trigonometry by Lardner, Lefebure de Fourcy, and Snowball, and the Treatise on Geometry published in the Library of Useful Knowledge. The examples have been chiefly selected from the University and College Examination Papers.

In the account of Napier's Rules of Circular Parts an explanation has been given of a method of proof devised by Napier, which seems to have been overlooked by most modern writers on the subject. I have had the advantage of access to an unprinted Memoir on this point by the late R. L. Ellis, of Trinity College; Mr. Ellis had in fact rediscovered for himself Napier's own method. For the use of this Memoir and for some valuable references on the subject I am indebted to the Dean of Ely.

Considerable labour has been bestowed on the text in order to render it comprehensive and accurate, and the examples have all been carefully verified; and thus I venture to hope that the work will be found useful by Students and Teachers.

I. TODHUNTER.

ST. JOHN'S COLLEGE,
August 15, 1859.

REVISER'S PREFACE.

In the present revision of Dr. Todhunter's *Spherical Trigonometry* so many changes have been made that only a comparatively small portion of the last edition remains in its original form. The introductory chapter, and the chapters on Geodetical Operations and on Polyhedrons, are almost untouched, and in the chapter on Arcs Drawn to Fixed Points only one paragraph has been altered. But that part of the book which deals with the Formulae of the Triangle and the Solution of Triangles has been re-written, and the remaining chapters include extensive alterations and additions.

I have followed the example of the late Dr. Casey in introducing chapters on Spherical Geometry, and I am indebted to his *Spherical Trigonometry*, and to Baltzer's *Elemente der Mathematik*, for references to the important writings on the subject. Passing over, however, a number of geometrical methods of considerable interest but of restricted application, I have given the central place in the present edition to the Principle of Duality as exemplified in theorems relating to circles on the sphere. Though the principle and some of its applications to Spherical Geometry have been known for

a long time, I have not found any connected account of the subject, such as is contained in Chapter X.

Coaxal circles have been discussed in such a way as to shew their analogy with coaxal circles on a plane; and the coaxal system and the reciprocal of a coaxal system, to which I have given the name *colunar*, are selected as examples of Duality, partly because the properties of the latter afford a new treatment of HART'S Theorem, but chiefly because, on transition to the plane, they present an interesting relation between systems of circles on the plane, possessed in the one case of a common radical axis, in the other of a common centre of similitude *

A chapter has been devoted to the generalisation of the Spherical Triangle, based on a recent memoir by Dr E. STUDY; and another gives a brief account of Prof. FROBENIUS'S application of determinants to the geometry of the sphere.

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* In this connexion a remark, which it is now too late to insert in its natural place in the text, may be made here.

Just as the constant of Art. 169 is called the Spherical Power of the point with respect to the small circle, so the constant of Art. 171 may be called the Spherical Power of the great circle with respect to the small circle. (If the great and the small circle intersect at an angle ϕ, the spherical power is equal to $\tan^2\frac{1}{2}\phi$.) Then, as the radical circle of two small circles is the *locus of points* whose spherical powers with respect to them are equal, the centre of similitude of two small circles is the *envelope of great circles* whose spherical powers with respect to them are equal. Of course by *the* centre of similitude of two circles is meant the external or the internal centre of similitude, according as the circles have the same or opposite senses of rotation assigned to them. This view of centres of similitude completes the analogy between coaxal and colunar circles, whether on a sphere or on a plane.

In both these chapters special attention has been paid to the conventions used for the purpose of avoiding ambiguity. It is hoped that a sufficient emphasis has thus been laid on the importance of assigning to every circle a certain direction and a unique pole, a method whose utility has been exemplified also in Chapter X.

Some examples have been added, taken, for the most part, from the papers of the Science and Art Examinations and of the Royal University of Ireland; a few are from REIDT'S collection.

My very grateful acknowledgments are due to Mr. T. J. I'ANSON BROMWICH, for his help in reading proofs, and for many most valuable suggestions.

J. G. LEATHEM.

ST. JOHN'S COLLEGE,
October 24, 1901.

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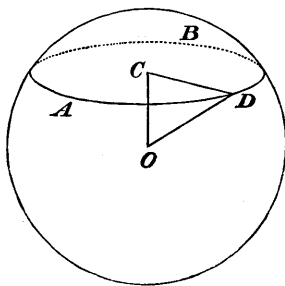
SPHERICAL TRIGONOMETRY.

CHAPTER I.

GREAT AND SMALL CIRCLES.

1. Definition. A *sphere* is a solid bounded by a surface every point of which is equally distant from a fixed point which is called the *centre* of the sphere. The straight line which joins any point of the surface with the centre is called a *radius*. A straight line drawn through the centre and terminated both ways by the surface is called a *diameter*.

2. *The section of the surface of a sphere made by any plane is a circle.*

Let AB be the section of the surface of a sphere made by any plane, O the centre of the sphere. Draw OC perpendicular

to the plane; take any point D in the section and join OD, CD. Since OC is perpendicular to the plane, the angle OCD is a right angle; therefore $CD = \sqrt{(OD^2 - OC^2)}$. Now O and C are fixed points, so that OC is constant; and OD is constant, being the radius of the sphere; hence CD is constant. Thus all points in the plane section are equally distant from the fixed point C; therefore the section is a circle of which C is the centre.

3. Definitions. The section of the surface of a sphere by a plane is called a *great circle* if the plane passes through the centre of the sphere, and a *small circle* if the plane does not pass through the centre of the sphere. Thus the radius of a great circle is equal to the radius of the sphere.

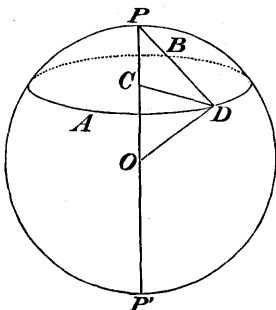
4. Through the centre of a sphere and any two points on the surface a plane can be drawn; and only one plane can be drawn, except when the two points are the extremities of a diameter of the sphere, and then an infinite number of such planes can be drawn. Hence only one great circle can be drawn through two given points on the surface of a sphere, except when the points are the extremities of a diameter of the sphere. When only one great circle can be drawn through two given points, the great circle is unequally divided at the two points; we shall for brevity speak of the shorter of the two arcs as *the* arc of a great circle joining the two points.

5. Definitions. The *axis* of any circle of a sphere is that diameter of the sphere which is perpendicular to the plane of the circle; the extremities of the axis are called the *poles** of the circle. The poles of a great circle are equally distant from the plane of the circle. The poles of a small circle are

* The expression *pole of a circle* is used by ARCHIMEDES (287-212 B.C.)

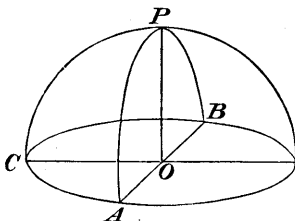
not equally distant from the plane of the circle; they may be called respectively the *nearer* and the *further* pole; sometimes the nearer pole is for brevity called *the* pole.

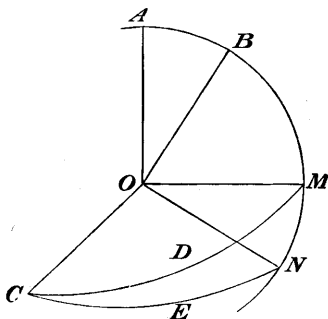
6. *A pole of a circle is equally distant from every point of the circumference of the circle.*

Let O be the centre of the sphere, AB any circle of the sphere, C the centre of the circle, P and P′ the poles of the circle. Take any point D in the circumference of the circle; join CD, OD, PD. Then $PD = \sqrt{(PC^2 + CD^2)}$; and PC and CD are constant, therefore PD is constant. Suppose a great circle to pass through the points P and D; then the chord PD is constant, and therefore the arc of a great circle intercepted between P and D is constant for all positions of D on the circle AB.

Thus the distance of a pole of a circle from every point of the circumference of the circle is constant, whether that distance be measured by the straight line joining the points, or by the arc of a great circle intercepted between the points.

Definition. The length of the arc, measured along a great circle, from any point on a small circle to the nearer pole is called the *spherical radius* of the small circle.

7. *The arc of a great circle which is drawn from a pole of a great circle to any point in its circumference is a quadrant.*

Let P be a pole of the great circle ABC; then the arc PA is a quadrant.

For let O be the centre of the sphere, and draw PO. Then PO is at right angles to the plane ABC, because P is the pole of ABC; therefore $P\hat{O}A$ is a right angle, and the arc PA is a quadrant.

8. *The angle subtended at the centre of a sphere by the arc of a great circle which joins the poles of two great circles is equal to the inclination of the planes of the great circles.*

Let O be the centre of the sphere, CD, CE the great circles intersecting at C, A and B the poles of CD and CE respectively.

Draw a great circle through A and B, meeting CD and CE at M and N respectively. Then AO is perpendicular to OC, which is a straight line in the plane OCD; and BO is perpendicular to OC, which is a straight line in the plane OCE; therefore OC is perpendicular to the plane AOB (Euclid, XI, 4); and therefore OC is perpendicular to the straight lines OM and ON, which are in the plane AOB. Hence $M\hat{O}N$ is the angle of inclination of the planes OCD and OCE. And

$$A\hat{O}B = A\hat{O}M - B\hat{O}M = B\hat{O}N - B\hat{O}M = M\hat{O}N.$$

9. Definition. When two circles intersect, the angle between the tangents at either of their points of intersection is called the angle between the circles.

The angle of intersection of two great circles is equal to the inclination of their planes.

For, in the figure of the preceding Article, the tangents at C to the circles CD and CE, lying in the planes of these circles respectively, are perpendicular to their common radius OC, which is the line of intersection of the planes. Hence the angle between the tangents is the angle of inclination of the planes.

In the figure of Art. 6, since PO is perpendicular to the plane ACB, every plane which contains PO is at right angles to the plane ACB. Hence the angle between the plane of any circle and the plane of a great circle which passes through its poles is a right angle.

10. *Two great circles bisect each other.*

For since the plane of each great circle passes through the centre of the sphere, the line of intersection of these planes is a diameter of the sphere, and therefore also a diameter of each great circle; therefore the great circles are bisected at the points where they meet.

11. *If the arcs of great circles joining a point* P *on the surface of a sphere with two other points* A *and* C *on the surface of the sphere, which are not at opposite extremities of a diameter, be each of them equal to a quadrant,* P *is a pole of the great circle through* A *and* C. (See the figure of Art. 7.)

For suppose PA and PC to be quadrants, and O the centre of the sphere; then since PA and PC are quadrants, the angles POC and POA are right angles. Hence PO is at right angles to the plane AOC, and P is a pole of the great circle AC.

12. Definition. Great circles which pass through the poles of a great circle are called *secondaries* to that circle. Thus, in the figure of Art. 8, the point C is a pole of ABMN, and therefore CM and CN are parts of secondaries to ABMN. And the angle between CM and CN is measured by MN; that is, *the angle between any two great circles is measured by the arc they intercept on the great circle to which they are secondaries.*

13. *If from a point on the surface of a sphere there can be drawn two arcs of great circles, not parts of the same great circle, the planes of which are at right angles to the plane of a given circle, that point is a pole of the given circle.*

For, since the planes of these arcs are at right angles to the plane of the given circle, the line in which they intersect is perpendicular to the plane of the given circle, and is therefore the axis of the given circle; hence the point from which the arcs are drawn is a pole of the circle.

14. *To compare the arc of a small circle subtending any angle at the centre of the circle with the arc of a great circle subtending the same angle at its centre.*

Let ab be the arc of a small circle, C the centre of the circle, P the pole of the circle, O the centre of the sphere. Through P draw the great circles PaA and PbB, meeting the great circle of which P is a pole at A and B respectively; draw Ca, Cb, OA,

OB. Then Ca, Cb, OA, OB are all perpendicular to OP, because the planes aCb and AOB are perpendicular to OP; therefore

Ca is parallel to OA, and Cb is parallel to OB. Therefore the angle aCb = the angle AOB (Euclid, XI, 10). Hence,

$$\frac{\text{arc } ab}{\text{radius C}a} = \frac{\text{arc AB}}{\text{radius OA}} \text{ (\textit{Plane Trigonometry}, Art. 18);}$$

therefore,
$$\frac{\text{arc } ab}{\text{arc AB}} = \frac{\text{C}a}{\text{OA}} = \frac{\text{C}a}{\text{O}a} = \sin \text{P}\hat{\text{O}}a.$$

CHAPTER II.

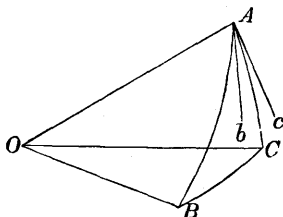
SPHERICAL TRIANGLES.

15. Spherical Trigonometry investigates the relations which subsist between the angles of the plane faces which form a solid angle and the angles at which the plane faces are inclined to each other.

16. Spherical Triangle. Suppose that the angular point of a solid angle is made the centre of a sphere; then the planes which form the solid angle will cut the sphere in arcs of great circles. Thus a figure will be formed on the surface of the sphere, which is called a *spherical triangle* if it is bounded by *three* arcs of great circles; this will be the case when the solid angle is formed by the meeting of *three* plane angles. If the solid angle be formed by the meeting of *more than three* plane angles, the corresponding figure on the surface of the sphere is bounded by more than three arcs of great circles, and is called a *spherical polygon.*

17. Definitions. The three arcs of great circles which form a spherical triangle are called the *sides* of the spherical triangle; the angles formed by the arcs at the points where they meet are called the *angles* of the spherical triangle. (See Art. 9.)

18. Thus, let O be the centre of a sphere, and suppose a solid angle formed at O by the meeting of three plane angles.

Let AB, BC, CA be the arcs of great circles in which the planes cut the sphere; then ABC is a spherical triangle, and the arcs AB, BC, CA are its sides. Suppose A*b* the tangent at A to the arc AB, and A*c* the tangent at A to the arc AC, the tangents

being drawn from A *towards* B and C respectively; then the angle *b*A*c* is one of the angles of the spherical triangle. Similarly angles formed in like manner at B and C are the other angles of the spherical triangle.

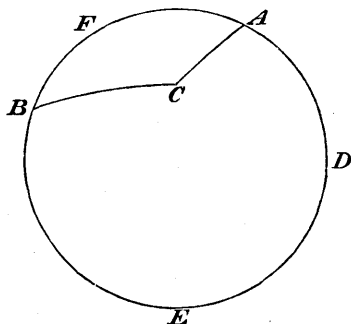
19. The principal part of a treatise on Spherical Trigonometry consists of theorems relating to spherical triangles; it is therefore necessary to obtain an accurate conception of a spherical triangle and its parts.

It will be seen that what are called *sides* of a spherical triangle are really *arcs* of great circles, and these arcs are proportional to the three plane angles which form the solid angle corresponding to the spherical triangle. Thus, in the figure of the preceding Article, the arc AB forms one side of the spherical triangle ABC, and the plane angle AOB is measured by the fraction $\frac{\text{arc AB}}{\text{radius OA}}$; and thus the arc AB is proportional to the angle AOB so long as we keep to the same sphere.

And from the proposition proved in Article 9 it follows that the *angles* of a spherical triangle are the same as the inclinations of the plane faces that form the solid angle.

20. Notation. The letters A, B, C are generally used to denote the *angles* of a spherical triangle, and the letters a, b, c are used to denote the *sides*. As in the case of plane triangles, A, B, and C may be used to denote the numerical values of the angles expressed in *terms of any unit*, provided we understand distinctly what the unit is. Thus, if the angle C be a right angle, we may say that $C = 90°$, or that $C = \frac{1}{2}\pi$, according as we adopt for the unit a degree or the angle subtended at the centre of a circle by an arc equal to the radius. So also, as the sides of a spherical triangle are proportional to the angles subtended at the centre of the sphere, we may use a, b, c to denote the numerical values of those angles in terms of any unit. We shall usually suppose both the angles and the sides of a spherical triangle to be expressed in *circular measure*. (*Plane Trigonometry*, Art. 20.)

21. In future, unless the contrary be distinctly stated, any arc drawn on the surface of a sphere will be supposed to be an arc of a *great* circle.

22. Conventional restriction of lengths of sides.* In

spherical triangles each side is restricted to be less than a

* See Chapter XIX.

semicircle; this is of course a *convention*, and it is adopted, for the present, partly because it is traditional and partly because it simplifies the study of Spherical Trigonometry for the beginner.

Thus, in the figure, the arc ADEB is greater than a semicircumference, and we might, if we pleased, consider ADEB, AC, and BC as forming a triangle, having its angular points at A, B, and C. But we agree to exclude such triangles from our consideration; and the triangle having its angular points at A, B, and C, will be understood to be that formed by AFB, BC, and CA.

23. From the restriction of the preceding Article it will follow that *any angle of a spherical triangle is less than two right angles.*

For suppose a triangle formed by BC, CA, and BEDA, having the angle BCA greater than two right angles. Then the parts of the arc BEDA, which are in the immediate neighbourhoods of B and of A respectively, clearly lie on opposite sides of the plane of the great circle BC. Hence the arc must cut this plane; let it do so in a point D; then D lies on the arc BC produced. By Art. 10, BED is a semicircle, and therefore BEA is greater than a semicircle; thus the proposed triangle is not one of those which we consider.

24. The relations between the sides and angles of a Spherical Triangle, which are investigated in treatises on Spherical Trigonometry, are chiefly such as involve the *Trigonometrical Functions* of the sides and angles. Before proceeding to these, however, we shall consider some theorems which involve the sides and angles *themselves*, and not their trigonometrical ratios.

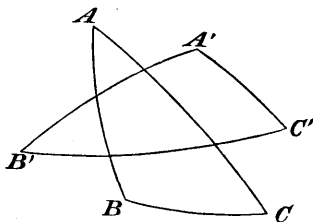
Definitions. The following definitions are important.

A *lune* is that portion of the surface of a sphere which is comprised between two great semicircles.

Two triangles, ABC, A″BC, which have a side BC common, and whose other sides belong to the same great circles, are called *colunar triangles*, as they together make up a lune. A″ is the point diametrically opposite to A on the sphere.

If A″, B″, C″ be diametrically opposite to A, B, C respectively, the triangle ABC has three colunar triangles, namely, A″BC, B″CA, and C″AB.

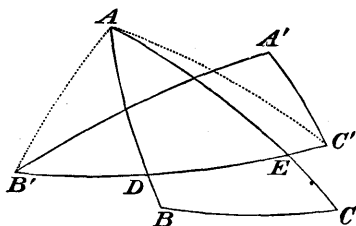
Antipodal triangles are triangles whose respective vertices are diametrically opposite to one another in pairs; such, for example, are the triangles ABC, A″B″C″.

25. Polar triangle.* Let ABC be any spherical triangle, and let the points A′, B′, C′ be those poles of the arcs BC, CA, AB respectively which lie on the same sides of them as the opposite angles A, B, C; then the triangle A′B′C′ is said to be the *polar triangle* of the triangle ABC.

Since there are two poles for each side of a spherical triangle, *eight* triangles can be formed having for their angular points poles of the sides of the given triangle; but there is only one triangle in which these poles A′, B′, C′ lie towards the same parts with the corresponding angles A, B, C; and this is the triangle which is known under the name of the *polar triangle*.

* The discovery of the polar triangle is due to SNELLIUS. Its use is explained in his *Trigonometria*, (Lib. III, Prop. VIII), published at Leyden in 1627.

The triangle ABC is called the *primitive* triangle with respect to the triangle A′B′C′.

26. *If one triangle be the polar triangle of another, the latter will be the polar triangle of the former.*

Let ABC be any triangle, A′B′C′ the polar triangle: then ABC will be the polar triangle of A′B′C′.

For since B′ is a pole of AC, the arc AB′ is a quadrant, and since C′ is a pole of BA, the arc AC′ is a quadrant (Art. 7); therefore A is a pole of B′C′ (Art. 11). Also A and A′ are on the same side of B′C′; for A and A′ are by hypothesis on the same side of BC, therefore A′A is less than a quadrant; and since A is a pole of B′C′, and AA′ is less than a quadrant, A and A′ are on the same side of B′C′.

Similarly it may be shewn that B is a pole of C′A′, and that B and B′ are on the same side of C′A′; also that C is a pole of A′B′, and that C and C′ are on the same side of A′B′. Thus ABC is the polar triangle of A′B′C′.

27. *The sides and angles of the polar triangle are respectively the supplements of the angles and sides of the primitive triangle.*

For let the arc B′C′, produced if necessary, meet the arcs AB, AC, produced if necessary, at the points D and E respectively; then since A is a pole of B′C′, the spherical angle A is measured by the arc DE (Art. 12). But B′E and C′D are each quadrants; therefore DE and B′C′ are together equal to a semicircle; that

is, the angle subtended by B′C′ at the centre of the sphere is the supplement of the angle A. This we may express for shortness thus; B′C′ is the supplement of A. Similarly it may be shewn that C′A′ is the supplement of B, and A′B′ the supplement of C.

And since ABC is the polar triangle of A′B′C′, it follows that BC, CA, AB are respectively the supplements of A′, B′, C′; that is, A′, B′, C′ are respectively the supplements of BC, CA, AB.

From these properties a primitive triangle and its polar triangle are sometimes called *supplemental triangles.*

Thus, if A, B, C, a, b, c denote respectively the angles and the sides of a spherical triangle, all expressed in circular measure, and A′, B′, C′, a', b', c' those of the polar triangle, we have

$$A' = \pi - a, \quad B' = \pi - b, \quad C' = \pi - c,$$

$$a' = \pi - A, \quad b' = \pi - B, \quad c' = \pi - C.$$

28. Duality of theorems relating to the spherical triangle. The preceding result is of great importance; for if any general theorem be demonstrated with respect to the sides and the angles of any spherical triangle it holds of course for the polar triangle also. *Thus any such theorem will remain true when the angles are changed into the supplements of the corresponding sides and the sides into the supplements of the corresponding angles.* We shall see several examples of this principle in the next Chapter.

29. *Any two sides of a spherical triangle are together greater than the third side.* (See the figure of Art. 18.)

For any two of the three plane angles which form the solid angle at O are together greater than the third (Euclid, XI, 20). Therefore any two of the arcs BC, CA, AB, are together greater than the third.

From this proposition it is obvious that any side of a spherical triangle is greater than the difference of the other two.

30. *The sum of the three sides of a spherical triangle is less than the circumference of a great circle.* (See the figure of Art. 18.)

For the sum of the three plane angles which form the solid angle at O is less than four right angles (Euclid, XI, 21);

therefore, $\frac{BC}{OA} + \frac{CA}{OA} + \frac{AB}{OA}$ is less than 2π,

therefore, $BC + CA + AB$ is less than $2\pi \times OA$;

that is, the sum of the arcs is less than the circumference of a great circle.

31. The propositions contained in the preceding two Articles may be extended. Thus, if there be any polygon which has each of its angles less than two right angles, *any one side is less than the sum of all the others.* This may be proved by repeated use of Art. 29. Suppose, for example, that the figure has four sides, and let the angular points be denoted by A, B, C, D.

Then $AB + BC$ is greater than AC;

therefore, $AB + BC + CD$ is greater than $AC + CD$,

and *à fortiori* greater than AD.

Again, if there be any polygon which has each of its angles less than two right angles, *the sum of its sides will be less than the circumference of a great circle.* This follows from Euclid, XI, 21, in the manner shewn in Art. 30.

32. *The three angles of a spherical triangle are together greater than two right angles, and less than six right angles.*

Let A, B, C be the *angles* of a spherical triangle; let a', b', c', be the *sides* of the polar triangle. Then by Art. 30,

$a' + b' + c'$ is less than 2π,

that is, $\pi - A + \pi - B + \pi - C$ is less than 2π;

therefore $A + B + C$ is greater than π.

And since each of the angles A, B, C is less than π, the sum $A + B + C$ is less than 3π.

33. Identical and symmetrical equality of triangles. If ABC, A″B″C″ be antipodal triangles, the plane of the arc BC is the same as the plane of the arc B″C″, and similarly for CA, C″A″ and AB, A″B″. Hence the angles of the one triangle are respectively equal to those of the other; and as the distance between two points is equal to the distance between the diametrically opposite points, the sides of one triangle are equal to the corresponding sides of the other. Thus the triangles have all their corresponding elements equal.

There is however this difference between them, that if we go round the two triangles in such a manner as to take corresponding elements in the same order, we shall go round one triangle in the clockwise, the other in the counter-clockwise sense. And so if the triangle A″B″C″ be shifted bodily in the surface of the sphere until B″ coincides with B, and C″ with C, then the remaining vertices A″ and A will not coincide, but will be on opposite sides of the common arc BC. The triangles therefore are not superposable. If however the triangle A″B″C″, regarded as a material film, were lifted off the sphere and, as it were, turned inside out, so that the formerly convex side of its surface would become concave, the altered triangle could then be exactly superposed on the triangle ABC.

Antipodal triangles are accordingly equal to one another in every respect, and yet not superposable in the ordinary meaning of the term. Triangles having this sort of equality are said to be *symmetrically** equal, as distinguished from triangles which are superposable and which are said to be *identically* equal, or *congruent*.

34. The proof, by the method of superposition, of the equality of plane triangles under certain circumstances, as used for example in Euclid, I, 4, 8, and 26, may be applied equally well to spherical triangles on the same sphere; the

* This term is due to LEGENDRE (*Géométrie*, VI, Def. 16.)

test of equality being that one triangle should be superposable on the other, or on the antipodal triangle of the other. In this way may be proved the first three cases of the following theorem :

Two triangles on the same sphere are either congruent or symmetrically equal, and therefore have all their corresponding elements equal,

(1) *When two sides and the included angle of one are respectively equal to two sides and the included angle of the other.*

(2) *When the three sides of one are respectively equal to the three sides of the other.*

(3) *When two angles and the adjacent side of one are respectively equal to two angles and the adjacent side of the other.*

(4) *When the three angles of one are respectively equal to the three angles of the other.*

Case (4) has no analogue in plane geometry; it is derived from Case (2) by consideration of the supplemental triangles.

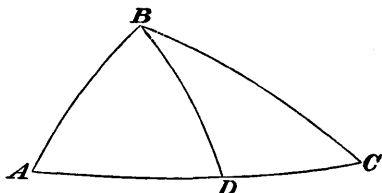
35. *The angles at the base of an isosceles spherical triangle are equal.*

For if the sides AB, AC are equal, and if D be the mid point of BC, the triangles ADB, ADC have their corresponding sides equal each to each, and therefore are symmetrically equal. Hence the angles B and C are equal.

If AB and AC are quadrants, the angles at the base are right angles by Arts. 11 and 9.

36. *If two angles of a spherical triangle are equal, the opposite sides are equal.*

Since the primitive triangle has two equal angles, the polar triangle has two equal sides; therefore in the polar triangle the angles opposite the equal sides are equal by Art. 35. Hence in the primitive triangle the sides opposite the equal angles are equal.

37. *If one angle of a spherical triangle is greater than another, the side opposite the greater angle is greater than the side opposite the less angle.*

Let ABC be a spherical triangle, and let the angle ABC be greater than the angle BAC: then the side AC will be greater than the side BC. At B make the angle ABD equal to the angle BAD; then BD is equal to AD (Art. 36), and BD + DC is greater than BC (Art. 29); therefore AD + DC is greater than BC; that is, AC is greater than BC.

38. *If one side of a spherical triangle is greater than another, the angle opposite the greater side is greater than the angle opposite the less side.*

This follows from the preceding Article by means of the polar triangle.

Or thus; suppose the side AC greater than the side BC, then the angle ABC will be greater than the angle BAC. For the angle ABC cannot be less than the angle BAC by Art. 37, and the angle ABC cannot be equal to the angle BAC by Art. 36; therefore the angle ABC must be greater than the angle BAC.

This Chapter might be extended; but it is unnecessary to do so, because the Trigonometrical formulae of the next Chapter supply an easy method of investigating the theorems of Spherical Geometry. See, for example, Arts. 67 and 68.

39. Note.—The foundation of the science of Spherical Trigonometry is attributed to the astronomer Hipparchus (150 B.C.). Fundamental theorems of the subject are found in the *Sphaerica* of Menelaus and in

the *Almagest* of PTOLEMY. These were afterwards elaborated by the Arabs, and in the middle of the fifteenth century by REGIOMONTANUS, for use in Astronomy.

In modern times the study of Spherical Trigonometry received a fresh impetus from the writings of EULER, who published several memoirs on the subject. The first appeared in the *Mémoires de l'Académie Royale de Berlin* in 1753, and was followed some years later by a series of papers in the *Acta Petropolitana*; of these the most important are those entitled "*De Mensura Angulorum Solidorum*" (1778, p. 31), and "*Trigonometria Sphaerica Universa ex primis principiis derivata*" (1779, p. 79). LAGRANGE gave an investigation of the formulae of the spherical triangle a few years later in the *Journal de l'École Polytechnique* (1799, *Cahier* 6, p. 270).

The chief contributors to the science of Spherical Geometry, in addition to those already named, are VIETA (1595), NAPIER (1614), SNELLIUS (1626), GIRARD (1629), LEXELL (1782), LEGENDRE (1787), CHASLES (1831), SCHULZ (1833), GUDERMANN (1835), and BORGNET (1847).

The extension of the standard formulae to triangles whose sides and angles are not necessarily less than 180° is generally ascribed to MÖBIUS ("*Entwickelungen der Grundformeln der sphärischen Trigonometrie in grösstmöglicher Allgemeinheit*," *Verhandlungen der kön. säch. Gesellschaft der Wissenschaften zu Leipzig*, 1860, p. 51). But from a remark of GAUSS's in § 54 of his *Theoria Motus Corporum Coelestium* (1809), it is plain that he had thought of this generalisation, and had worked it out, though he did not publish it. Professor CHAUVENET, in the preface to his work on *Astronomy*, points out that in his own *Treatise on Trigonometry*, published in 1850 (a work at present out of print and difficult to procure), the standard formulae are proved for the general triangle; however, MÖBIUS's first memoir on the subject appeared in 1846 (cf. § 302). We shall discuss the generalisation of the triangle in Chapter XIX.

CHAPTER III.

RELATIONS BETWEEN THE TRIGONOMETRICAL FUNCTIONS OF THE SIDES AND THE ANGLES OF A SPHERICAL TRIANGLE.

40. Elements of a spherical triangle. A spherical triangle has six elements, namely the three sides a, b, c, and the three angles A, B, C. When any three of these are given, the form and dimensions of the triangle are completely determined, as there exist relations by means of which the other three elements can then be found. These relations will be established in the present chapter.

They may conveniently be divided into two classes. In Section I. will be considered those formulae which involve four elements of the triangle; in Section II. those which involve five or six elements.

SECTION I.

41. Formulae involving four elements. There are four cases, according as the elements involved are:

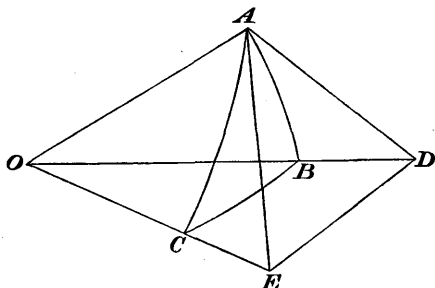
I. Three sides and an angle.

II. Two sides and the angles opposite to them.

III. Two sides, the included angle, and another angle.

IV. Three angles and a side.

Case. I.—Three sides and an angle.

42. *To express the cosine of an angle of a triangle in terms of sines and cosines of the sides.*

Let ABC be a spherical triangle, O the centre of the sphere. Let the tangent at A to the arc AC meet OC produced at E, and let the tangent at A to the arc AB meet OB produced at D; join ED. Thus the angle EAD is the angle A of the spherical triangle, and the angle EOD measures the side a.

From the triangles ADE and ODE we have

$$DE^2 = AD^2 + AE^2 - 2AD.AE\cos A,$$

$$DE^2 = OD^2 + OE^2 - 2OD.OE\cos a;$$

also the angles OAD and OAE are right angles, so that $OD^2 = OA^2 + AD^2$ and $OE^2 = OA^2 + AE^2$. Hence by subtraction we have

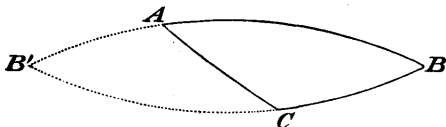
$$0 = 2OA^2 + 2AD.AE\cos A - 2OD.OE\cos a;$$

therefore $$\cos a = \frac{OA}{OE}\cdot\frac{OA}{OD} + \frac{AE}{OE}\cdot\frac{AD}{OD}\cos A;$$

that is $$\cos a = \cos b\cos c + \sin b\sin c\cos A; \quad \text{.............(1)}$$

therefore $$\cos A = \frac{\cos a - \cos b\cos c}{\sin b\sin c}. \quad \text{...............(2)}$$

43. We have supposed, in the construction of the preceding Article, that the sides which contain the angle A are less than quadrants, for we have assumed that the tangents at A meet OB and OC respectively produced. We must now shew that the formula obtained is true when these sides are not less than quadrants. This we shall do by special examination of the cases in which one side or each side is greater than a quadrant or equal to a quadrant.

(1) Suppose only one of the sides which contain the angle A to be greater than a quadrant, for example AB. Produce BA and BC to meet at B′; and put $AB' = c'$, $CB' = a'$.

Then we have from the triangle AB′C, by what has already been proved,

$$\cos a' = \cos b \cos c' + \sin b \sin c' \cos B'AC ;$$

but $a' = \pi - a$, $c' = \pi - c$, $B'AC = \pi - A$; thus

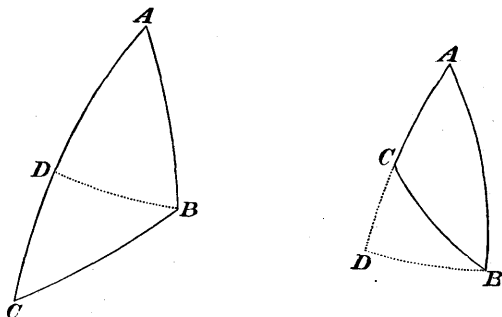
$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

(2) Suppose both the sides which contain the angle A to be greater than quadrants. Produce AB and AC to meet at A′; put $A'B = c'$, $A'C = b'$; then from the triangle A′BC, as before,

$$\cos a = \cos b' \cos c' + \sin b' \sin c' \cos A' ;$$

but $b' = \pi - b$, $c' = \pi - c$, $A' = A$; thus

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

(3) Suppose that one of the sides which contain the angle A is a quadrant, for example AB; on AC, produced if necessary, take AD equal to a quadrant and draw BD. If BD is a quadrant, B is a pole of

AC (Art. 11); in this case $a=\frac{1}{2}\pi$ and $A=\frac{1}{2}\pi$ as well as $c=\frac{1}{2}\pi$. Thus the formula to be verified reduces to the identity $0=0$ If BD is not a quadrant, the triangle BDC gives

$$\cos a=\cos CD \cos BD+\sin CD \sin BD \cos C\hat{D}B,$$

and $\cos C\hat{D}B=0, \quad \cos CD=\cos(\frac{1}{2}\pi \backsim b)=\sin b, \quad \cos BD=\cos A;$

thus $$\cos a=\sin b \cos A;$$

and this is what the formula in Art. 42 becomes when $c=\frac{1}{2}\pi$.

(4) Suppose that both the sides which contain the angle A are quadrants. The formula then becomes $\cos a=\cos A$; and this is obviously true, for A is now the pole of BC, and thus $A=a$.

Thus the formula in Art. 42 is proved to be universally true.

44. The formula in Art. 42 may be applied to express the cosine of *any* angle of a triangle in terms of sines and cosines of the sides; thus we have the three formulae,

$$\left.\begin{aligned}\cos a &= \cos b \cos c + \sin b \sin c \cos A,\\ \cos b &= \cos c \cos a + \sin c \sin a \cos B,\\ \cos c &= \cos a \cos b + \sin a \sin b \cos C.\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(3)^*$$

These may be considered as the fundamental equations of Spherical Trigonometry: we shall deduce various formulae from them.

45. *To express the sine of an angle of a spherical triangle in terms of trigonometrical functions of the sides.*

We have

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c};$$

therefore

$$\sin^2 A = 1 - \left(\frac{\cos a - \cos b \cos c}{\sin b \sin c}\right)^2$$

$$= \frac{(1 - \cos^2 b)(1 - \cos^2 c) - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$= \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c};$$

therefore

$$\sin A = \frac{\sqrt{(1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c)}}{\sin b \sin c}. \quad \ldots(4)$$

The radical on the right-hand side must be taken with the positive sign, because $\sin b$, $\sin c$, and $\sin A$ are all positive, owing to the restrictions of Arts. 22 and 23.

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* These formulae were discovered by ALBATEGNIUS, who made various applications of them. A demonstration of them is given by EULER (*Mémoires de Berlin*, 1753). All the other formulae of the spherical triangle may be deduced from them, as was shewn by LAGRANGE; GAUSS, also, in an appendix to SCHUMACHER'S translation of CARNOT'S *Géométrie de Position*, derives all the other formulae from them (GAUSS, *Ges. Werke*, vol. IV, p. 401).

Case II.—Two sides and the angles opposite to them.

46. From the value of $\sin A$ in the preceding Article it follows that

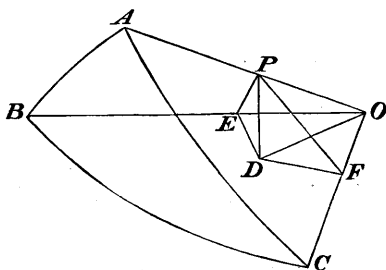
$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}, \quad \text{......................(5)}$$

for each of these is equal to the same expression, namely,

$$\frac{\sqrt{(1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c)}}{\sin a \sin b \sin c}. \quad \text{......(6)}$$

Thus *the sines of the angles of a spherical triangle are proportional to the sines of the opposite sides.* We shall give an independent proof of this proposition in the following Article.

47. *The sines of the angles of a spherical triangle are proportional to the sines of the opposite sides.**

Let ABC be a spherical triangle, O the centre of the sphere. Take any point P in OA, draw PD perpendicular to the plane BOC, and from D draw DE, DF perpendicular to OB, OC respectively; join PE, PF, OD.

Since PD is perpendicular to the plane BOC, it makes right angles with every straight line meeting it in that plane; hence

$$PE^2 = PD^2 + DE^2 = PO^2 - OD^2 + DE^2 = PO^2 - OE^2;$$

* This fundamental theorem of Spherical Trigonometry is found, under a rather different form, in the 3rd book of the *Sphaerica* of MENELAUS.

thus PEO is a right angle. Therefore $PE = OP \sin P\hat{O}E = OP \sin c$; and $PD = PE \sin P\hat{E}D = PE \sin B = OP \sin c \sin B$.

Similarly, $PD = OP \sin b \sin C$; therefore

$$OP \sin c \sin B = OP \sin b \sin C;$$

therefore $$\frac{\sin B}{\sin C} = \frac{\sin b}{\sin c}.$$

The figure supposes b, c, B, and C each less than a right angle; it will be found on examination that the proof will hold when the figure is modified to meet any case which can occur. If, for instance, B alone is greater than a right angle, the point D will fall beyond OB instead of between OB and OC; then $P\hat{E}D$ will be the *supplement* of B, and thus $\sin P\hat{E}D$ is still equal to $\sin B$.

Case III.—Two sides, the included angle, and another angle.

48. *To shew that* $\cot a \sin b = \cot A \sin C + \cos b \cos C$.

We have $$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C,$$
$$\sin c = \sin a \frac{\sin C}{\sin A}.$$

Substitute the values of $\cos c$ and $\sin c$ in the first equation; thus

$$\cos a = (\cos a \cos b + \sin a \sin b \cos C) \cos b + \frac{\sin a \sin b \cos A \sin C}{\sin A};$$

by transposition

$$\cos a \sin^2 b = \sin a \sin b \cos b \cos C + \sin a \sin b \cot A \sin C;$$

divide by $\sin a \sin b$; thus

$$\cot a \sin b = \cos b \cos C + \cot A \sin C. \quad \text{............}(7)$$

49. By interchanging the letters five other formulae, like that in the preceding Article, may be obtained; the whole six formulae will be as follows:

$$\left.\begin{aligned}\cot a \sin b &= \cot \mathrm{A} \sin \mathrm{C} + \cos b \cos \mathrm{C},\\ \cot b \sin a &= \cot \mathrm{B} \sin \mathrm{C} + \cos a \cos \mathrm{C},\\ \cot b \sin c &= \cot \mathrm{B} \sin \mathrm{A} + \cos c \cos \mathrm{A},\\ \cot c \sin b &= \cot \mathrm{C} \sin \mathrm{A} + \cos b \cos \mathrm{A},\\ \cot c \sin a &= \cot \mathrm{C} \sin \mathrm{B} + \cos a \cos \mathrm{B},\\ \cot a \sin c &= \cot \mathrm{A} \sin \mathrm{B} + \cos c \cos \mathrm{B}.\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(8)$$

Of the angles and sides entering into any one of these formulae, one of the angles is contained by the two sides and may be called the *inner* angle, and one of the sides lies between the two angles and may be called the *inner* side. The formula may then be stated thus :—

(cosine of inner side)(cosine of inner angle)
= (sine of inner side)(cotangent of other side)
− (sine of inner angle)(cotangent of other angle.)

This verbal expression of the formulae* is a convenient one to remember.

Other formulae of Case I.

50. *To express the sine, cosine, and tangent of half an angle of a triangle as functions of the sides.*

We have, by Art. 42, $\cos \mathrm{A} = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$;

therefore $1 - \cos \mathrm{A} = 1 - \dfrac{\cos a - \cos b \cos c}{\sin b \sin c} = \dfrac{\cos(b-c) - \cos a}{\sin b \sin c}$;

therefore $$\sin^2\frac{\mathrm{A}}{2} = \frac{\sin \frac{1}{2}(a+b-c) \sin \frac{1}{2}(a-b+c)}{\sin b \sin c}. \quad \ldots\ldots\ldots\ldots(9)$$

Let $2s = a+b+c$, so that s is half the sum of the sides of the triangle; then

$$a+b-c = 2s-2c = 2(s-c), \quad a-b+c = 2s-2b = 2(s-b);$$

thus $$\sin^2\frac{\mathrm{A}}{2} = \frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}, \quad \ldots\ldots\ldots\ldots(10)$$

* Suggested to the reviser by a friend.

and $$\sin\frac{A}{2} = \sqrt{\left\{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}\right\}} \qquad \text{.....(11)*}$$

Also, $$1 + \cos A = 1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c} = \frac{\cos a - \cos(b+c)}{\sin b \sin c};$$

therefore

$$\cos^2\frac{A}{2} = \frac{\sin\frac{1}{2}(a+b+c)\sin\frac{1}{2}(b+c-a)}{\sin b \sin c} = \frac{\sin s \sin(s-a)}{\sin b \sin c}, \qquad \text{...(12)}$$

and $$\cos\frac{A}{2} = \sqrt{\left\{\frac{\sin s \sin(s-a)}{\sin b \sin c}\right\}} \qquad \text{.....(13)*}$$

From the expressions for $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ we deduce

$$\tan\frac{A}{2} = \sqrt{\left\{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}\right\}} \qquad \text{.....(14)*}$$

The positive sign must be given to the radicals which occur in this Article, because $\frac{A}{2}$ is less than a right angle, and therefore its sine, cosine, and tangent are all positive.

51. Since $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$, we obtain

$$\sin A = \frac{2}{\sin b \sin c}\{\sin s \sin(s-a)\sin(s-b)\sin(s-c)\}^{\frac{1}{2}} \qquad \text{...(15)}$$

It may be shewn that the expression for $\sin A$ in Art. 45 agrees with the present expression, by putting the numerator of that expression in factors, as in *Plane Trigonometry*, Art. 115. We shall find it convenient to use a special symbol for the radical in the value of $\sin A$; we shall denote it by n, so that

$$n^2 = \sin s \sin(s-a)\sin(s-b)\sin(s-c), \qquad \text{.....(16)}$$

and $$4n^2 = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c. \qquad \text{...(17)†}$$

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* Euler, 1753.

† These expressions for n are given by Euler (*Novi Commentarii Petropolitani*, Vol. IV, p. 158).

The expression here represented by n occurs so frequently in Spherical Trigonometry that it is convenient to have a definite name for it. Professor NEUBERG of Liège suggests that it be called the *norm of the sides* of the triangle, and that the corresponding function of the supplements of the angles, usually represented by N, be called the *norm of the angles.* Professor von STAUDT * calls $2n$ the *sine* of the trihedral angle subtended by the triangle at the centre of the sphere, or more briefly the *sine of the triangle* †; in like manner 2N is the *sine of the polar triangle.*

52. Derivation of formulae by projection. The following compact method of obtaining the fundamental equations of Spherical Trigonometry is given in Col. A. R. CLARKE'S *Geodesy.* ‡ Depending, as it does, on the principles of geometrical projection, it holds equally well for all triangles, without any restriction on the magnitudes of the sides or angles.

Join O, the centre of the sphere, with the angular points A, B, C of the spherical triangle: let Q, R be the projections of C on OA and OB, P its projection on the plane AOB, S the projection of Q on OB.

Then
$$\text{OR} = \text{OS} + \text{QP} \cos (c - \tfrac{1}{2}\pi),$$
$$\text{RP} = \text{SQ} - \text{QP} \cos c,$$
$$\text{QC} \sin \text{A} = \text{PC} = \text{RC} \sin \text{B}.$$

Here make the following substitutions:

$$\begin{aligned} \text{OR} &= \cos a, & \text{OS} &= \cos b \cos c, \\ \text{RC} &= \sin a, & \text{QP} &= \sin b \cos \text{A}, \\ \text{OQ} &= \cos b, & \text{RP} &= \sin a \cos \text{B}, \\ \text{QC} &= \sin b, & \text{SQ} &= \cos b \sin c, \end{aligned}$$

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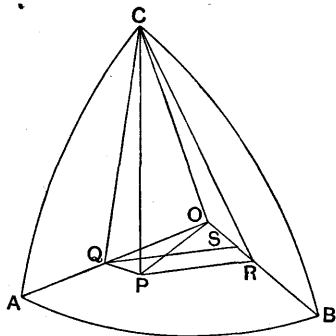
* *Crelle's Journal,* XXIV, 1842, p. 252.

† The area of a plane triangle is equal to half the product of two sides and the sine of the angle between them; and the volume of a tetrahedron is equal to one-sixth of the product of three conterminous edges and the sine of the trihedral angle between them. The analogy between these two results shews the reason of von STAUDT'S nomenclature.

‡ Compare GAUSS, *Theoria Motus,* § 59.

and we have immediately

$$\cos a = \cos b \cos c + \sin b \sin c \cos \mathsf{A}, \quad\ldots\ldots\ldots\ldots(18)$$

$$\sin a \cos \mathsf{B} = \cos b \sin c - \sin b \cos c \cos \mathsf{A}, \quad\ldots\ldots\ldots\ldots(19)$$

$$\sin a \sin \mathsf{B} = \sin b \sin \mathsf{A}. \quad\ldots\ldots\ldots\ldots(20)$$

The first and third of these are standard forms. If we multiply the second through by sin A, and then divide its sides by the corresponding sides of the third, we obtain the standard formula

$$\sin \mathsf{A} \cot \mathsf{B} = \sin c \cot b - \cos c \cos \mathsf{A}. \quad\ldots\ldots\ldots\ldots(21)$$

53. The formulae (18), (19), and (20) are analogous to the formulae

$$a^2 = b^2 + c^2 - 2bc \cos \mathsf{A}, \quad\ldots\ldots\ldots\ldots(18')$$

$$a \cos \mathsf{B} = c - b \cos \mathsf{A}, \quad\ldots\ldots\ldots\ldots(19')$$

$$a \sin \mathsf{B} = b \sin \mathsf{A}, \quad\ldots\ldots\ldots\ldots(20')$$

of the plane triangle, as will readily be seen on applying the method of Chapter XV. For this reason, and because they naturally present themselves in the proof by projection, these might well be regarded as the fundamental formulae of the spherical triangle. They are the forms used by CHAUVENET throughout his work on *Astronomy*, and are readily adapted to logarithms by putting

$$\cos b = r \cos \theta, \qquad \sin b \cos \mathsf{A} = r \sin \theta, \quad\ldots\ldots\ldots\ldots(22)$$

whence follow

$$\left.\begin{aligned} \cos a &= r \cos (c - \theta), \\ \sin a \cos \mathsf{B} &= r \sin (c - \theta). \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(23)$$

It has, however, always been customary in English treatises on Spherical Trigonometry to place (21) among the standard formulae instead of (19).

Case IV.—Three angles and a side.

54. *To express the cosine of a side of a triangle in terms of sines and cosines of the angles.*

In the formulae of Art. 42 we may, by Art. 28, change the sides into the supplements of the corresponding angles and the angle into the supplement of the corresponding side; thus

$$\cos(\pi - A) = \cos(\pi - B)\cos(\pi - C) + \sin(\pi - B)\sin(\pi - C)\cos(\pi - a),$$

that is,

$$\left.\begin{aligned} \cos A &= -\cos B\cos C + \sin B\sin C\cos a. \\ \text{Similarly}\quad \cos B &= -\cos C\cos A + \sin C\sin A\cos b, \\ \text{and}\quad \cos C &= -\cos A\cos B + \sin A\sin B\cos c. \end{aligned}\right\} \quad\ldots\ldots\ldots(24)^*$$

55. The formulae in Art. 49 will of course remain true when the angles and sides are changed into the supplements of the corresponding sides and angles respectively; it will be found, however, that no *new* formulae are thus obtained, but only the *same* formulae over again. This consideration will furnish some assistance in retaining those formulae accurately in the memory.

56. *To express the sine, cosine, and tangent, of half a side of a triangle as functions of the angles.*

We have, by Art. 54, $\cos a = \dfrac{\cos A + \cos B\cos C}{\sin B\sin C}$;

therefore

$$1 - \cos a = 1 - \frac{\cos A + \cos B\cos C}{\sin B\sin C} = -\frac{\cos A + \cos(B + C)}{\sin B\sin C};$$

therefore

$$\sin^2\frac{a}{2} = -\frac{\cos\frac{1}{2}(A + B + C)\cos\frac{1}{2}(B + C - A)}{\sin B\sin C}. \quad\ldots\ldots(25)$$

* These formulae were first published by VIETA in 1595, in the eighth book of his *Variorum de rebus mathematicis responsorum.*

Let $2S = A + B + C$; then $B + C - A = 2(S - A)$, therefore

$$\sin^2\frac{a}{2} = -\frac{\cos S \cos(S - A)}{\sin B \sin C}, \quad\ldots\ldots(26)$$

and

$$\sin\frac{a}{2} = \sqrt{\left\{-\frac{\cos S \cos(S - A)}{\sin B \sin C}\right\}}. \quad\ldots\ldots(27)$$

Also $1 + \cos a = 1 + \dfrac{\cos A + \cos B \cos C}{\sin B \sin C} = \dfrac{\cos A + \cos(B - C)}{\sin B \sin C}$;

therefore

$$\cos^2\frac{a}{2} = \frac{\cos\frac{1}{2}(A - B + C)\cos\frac{1}{2}(A + B - C)}{\sin B \sin C} = \frac{\cos(S - B)\cos(S - C)}{\sin B \sin C}, \quad(28)$$

and

$$\cos\frac{a}{2} = \sqrt{\left\{\frac{\cos(S - B)\cos(S - C)}{\sin B \sin C}\right\}}. \quad\ldots\ldots(29)$$

Hence

$$\tan\frac{a}{2} = \sqrt{\left\{-\frac{\cos S \cos(S - A)}{\cos(S - B)\cos(S - C)}\right\}}. \quad\ldots\ldots(30)$$

The positive sign must be given to the radicals which occur in this Article, because $\frac{a}{2}$ is less than a right angle.

The expressions of this Article may also be obtained immediately from those given in Art. 50 by the method of Art. 28.

57. It may be remarked that the values of $\sin\frac{1}{2}a$, $\cos\frac{1}{2}a$, and $\tan\frac{1}{2}a$ are *real*. For it has been shewn in Art. 32 that $2S$, the sum of the angles of the triangle, is greater than two right angles and less than six; hence S is greater than one right angle and less than three, and accordingly $\cos S$ is negative. Again, in the polar triangle, any side is less than the sum of the other two, and these sides are the supplements of A, B, C; thus $\pi - A$ is less than $\pi - B + \pi - C$; therefore $B + C - A$ is less than π, and consequently $S - A$ is less than $\frac{1}{2}\pi$. Also, as A cannot exceed π, $B + C - A$ is algebraically greater than $-\pi$, so that $S - A$ is algebraically greater than $-\frac{1}{2}\pi$. Thus $S - A$ lies between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, and therefore $\cos(S - A)$ is *positive*. Similarly, also, $\cos(S - B)$ and $\cos(S - C)$ are positive. Hence the expressions found above for $\sin^2\frac{1}{2}a$, $\cos^2\frac{1}{2}a$, and $\tan^2\frac{1}{2}a$ are positive, and have real square roots.

58. Since $\sin a = 2\sin\frac{a}{2}\cos\frac{a}{2}$, we obtain

$$\sin a = \frac{2}{\sin B \sin C}\{-\cos S\cos(S-A)\cos(S-B)\cos(S-C)\}^{\frac{1}{2}}. (31)$$

We shall use N to denote

$$\{-\cos S\cos(S-A)\cos(S-B)\cos(S-C)\}^{\frac{1}{2}}.*$$

59. The properties of supplemental triangles were proved geometrically in Art. 27, and by means of these properties the formulae in Art. 54 were obtained; but these formulae may be deduced analytically from those in Art. 44, and thus the whole subject may be made to depend on the formulae of Art. 44.

For from Art. 44 we obtain expressions for $\cos A$, $\cos B$, $\cos C$; and from these we find

$$\cos A + \cos B\cos C = \frac{(\cos a - \cos b\cos c)\sin^2 a + (\cos b - \cos a\cos c)(\cos c - \cos a\cos b)}{\sin^2 a\sin b\sin c}.$$

In the numerator of this fraction write $1-\cos^2 a$ for $\sin^2 a$; thus the numerator will be found to reduce to

$$\cos a(1-\cos^2 a - \cos^2 b - \cos^2 c + 2\cos a\cos b\cos c),$$

and this is equal to $\cos a\sin B\sin C\sin^2 a\sin b\sin c$ (Art. 46);

therefore $$\cos A + \cos B\cos C = \cos a\sin B\sin C.$$

Similarly the other two formulae of the same type may be proved.

Thus the formulae in Art. 54 are established; and therefore, without assuming the existence and properties of the Polar Triangle, we deduce the following theorem: *If the sides and angles of a spherical triangle be changed respectively into the supplements of the corresponding angles and sides, the fundamental formulae of Art. 44 hold good, and therefore also all results deducible from them.*

* The various expressions representing the value of N were obtained by Lexell (*Acta Petropolitana*, 1782, p. 49).

SECTION II.

Formulae involving five or six elements.

60. Napier's analogies.*

We have $$\frac{\sin A}{\sin a}=\frac{\sin B}{\sin b}=m \text{ suppose};\qquad\text{(32)}$$
and accordingly
$$\sin A+\sin B=m(\sin a+\sin b),\qquad\text{(33)}$$
$$\sin A-\sin B=m(\sin a-\sin b).\qquad\text{(34)}$$

Now $\cos A+\cos B\cos C=\sin B\sin C\cos a=m\sin C\sin b\cos a$,
and $\cos B+\cos A\cos C=\sin A\sin C\cos b=m\sin C\sin a\cos b$,
therefore, by addition,
$$(\cos A+\cos B)(1+\cos C)=m\sin C\sin(a+b);\qquad\text{(35)}$$
therefore by (33) we have
$$\frac{\sin A+\sin B}{\cos A+\cos B}=\frac{\sin a+\sin b}{\sin(a+b)}\,\frac{1+\cos C}{\sin C},$$
that is, $$\tan\tfrac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C.\qquad\text{(36)}$$

Similarly from (35) and (34) we have
$$\frac{\sin A-\sin B}{\cos A+\cos B}=\frac{\sin a-\sin b}{\sin(a+b)}\,\frac{1+\cos C}{\sin C},$$
that is, $$\tan\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C.\qquad\text{(37)}$$

By substituting in (36) and (37) the elements of the polar triangle, we obtain further
$$\tan\tfrac{1}{2}(a+b)=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c,\qquad\text{(38)}$$
$$\tan\tfrac{1}{2}(a-b)=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c.\qquad\text{(39)}$$

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* The formulae known by this name were discovered by NAPIER, and published by him in 1614 in his *Mirifici Logarithmorum canonis descriptio, ejusque usus in utraque trigonometria.*

The formulae (36), (37), (38), (39) may be put in the form of proportions or analogies, and are called from their discoverer NAPIER'S *Analogies*; the last two may be demonstrated without recurring to the polar triangle, by starting with the formulae of Art. 44.

61. In equation (36) of the preceding Article, $\cos\frac{1}{2}(a-b)$ and $\cot\frac{1}{2}\mathrm{C}$ are necessarily positive quantities; hence the equation shews that $\tan\frac{1}{2}(\mathrm{A}+\mathrm{B})$ and $\cos\frac{1}{2}(a+b)$ are of the same sign; thus $\frac{1}{2}(\mathrm{A}+\mathrm{B})$ and $\frac{1}{2}(a+b)$ are either both less than a right angle or both greater than a right angle. This is expressed by saying that $\frac{1}{2}(\mathrm{A}+\mathrm{B})$ and $\frac{1}{2}(a+b)$ *are of the same affection.*

62. Another proof of Napier's analogies. The proof of Article 60 starts from the fundamental formulae of the triangle. If use be made of the expressions for the tangents of the half angles in terms of the sides obtained in Art. 50, a shorter proof may be given.

For from these expressions it follows that

$$\tan\tfrac{1}{2}\mathrm{A}\tan\tfrac{1}{2}\mathrm{B}=\frac{\sin(s-c)}{\sin s}, \qquad\qquad (40)$$

and similarly for the other products of tangents. Hence, on substituting in the right-hand side of the identity

$$\tan\tfrac{1}{2}(\mathrm{A}+\mathrm{B})\tan\tfrac{1}{2}\mathrm{C}=\frac{\tan\frac{1}{2}\mathrm{B}\tan\frac{1}{2}\mathrm{C}+\tan\frac{1}{2}\mathrm{C}\tan\frac{1}{2}\mathrm{A}}{1-\tan\frac{1}{2}\mathrm{A}\tan\frac{1}{2}\mathrm{B}},$$

we get

$$\tan\tfrac{1}{2}(\mathrm{A}+\mathrm{B})\tan\tfrac{1}{2}\mathrm{C}=\frac{\sin(s-a)+\sin(s-b)}{\sin s-\sin(s-c)}=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}, \qquad (41)$$

and similarly

$$\tan\tfrac{1}{2}(\mathrm{A}-\mathrm{B})\tan\tfrac{1}{2}\mathrm{C}=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}. \qquad\qquad (42)$$

These correspond to results (36) and (37) above. Results

(38) and (39) may be obtained by a similar use of the expressions for the tangents of the half sides in terms of the angles.

63. Delambre's analogies. In the identity

$$\sin \tfrac{1}{2}(A+B) = \sin \tfrac{1}{2} A \cos \tfrac{1}{2} B + \cos \tfrac{1}{2} A \sin \tfrac{1}{2} B,$$

substitute the expressions obtained in Art. 50 for the sines and cosines of the half angles.

Thus, $$\sin \tfrac{1}{2}(A+B) = \frac{\sin(s-b)+\sin(s-a)}{\sin c}\sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}}$$

$$= \frac{\sin(s-b)+\sin(s-a)}{\sin c}\cos \tfrac{1}{2} C,$$

and so $$\frac{\sin \frac{1}{2}(A+B)}{\cos \frac{1}{2} C} = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2} c}. \quad \ldots\ldots(43)$$

In a precisely similar manner it may be shewn that

$$\frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2} C} = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2} c}, \quad \ldots\ldots(44)$$

$$\frac{\cos \frac{1}{2}(A+B)}{\sin \frac{1}{2} C} = \frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2} c}, \quad \ldots\ldots(45)$$

$$\frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2} C} = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2} c}. \quad \ldots\ldots(46)$$

These formulae are sometimes, but improperly, called GAUSS'S *Theorems*; they were first discovered by DELAMBRE. They were published almost simultaneously by GAUSS (*Theoria motus corporum coelestium*, § 54), DELAMBRE (*Connaissance des Tems*, 1809, p. 443), and MOLLWEIDE (ZACH'S *Monatliche Correspondenz*, 1808, p. 394). See the *Philosophical Magazine* for February, 1873.

64. NAPIER'S analogies may be derived from those of DELAMBRE simply by division. Thus if the sides of equation (46) be divided by the corresponding sides of equation (45), NAPIER'S third analogy is obtained.

DELAMBRE'S analogies may be derived from those of NAPIER as follows. Squaring NAPIER'S first analogy, (36), and adding unity to each side of the resulting equation, we get

$$\sec^2\tfrac{1}{2}(A+B) = \frac{\cos^2\frac{1}{2}(a-b)\cos^2\frac{1}{2}C + \cos^2\frac{1}{2}(a+b)\sin^2\frac{1}{2}C}{\cos^2\frac{1}{2}(a+b)\sin^2\frac{1}{2}C}. \quad \ldots\ldots(47)$$

The numerator of the right hand side is the same as

$$\tfrac{1}{2}\{1+\cos(a-b)\}\cos^2\tfrac{1}{2}C+\tfrac{1}{2}\{1+\cos(a+b)\}\sin^2\tfrac{1}{2}C,$$

or

$$\tfrac{1}{2}\{1+\cos a\cos b+\sin a\sin b\cos C\};$$

and this of course equals $\frac{1}{2}(1+\cos c)$.

Accordingly $$\sec^2\tfrac{1}{2}(A+B)=\frac{\cos^2\frac{1}{2}c}{\cos^2\frac{1}{2}(a+b)\sin^2\frac{1}{2}C}. \quad\text{.....................(48)}$$

Extracting the square roots, and determining the sign by the considera-tion that $\frac{1}{2}(A+B)$ and $\frac{1}{2}(a+b)$ are of the same affection (Art. 61), we obtain

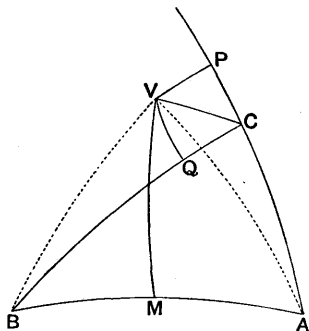
$$\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c=\cos\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C. \quad\text{..................(49)}$$

Similarly from NAPIER'S second analogy, (37),

$$\cos\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c=\sin\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C. \quad\text{..................(50)}$$

These are the third and fourth of DELAMBRE'S analogies; the other two are got by multiplying them respectively by the first two of NAPIER'S.

65. Geometrical proof of Delambre's and Napier's analogies.*

Bisect the side C at right angles by the arc MV, which meets the exterior bisector of the vertical angle C in V.

Draw the arcs VP, VQ perpendicular to the sides b, a of the triangle. Since VP = VQ, and VA = VB, and the angles at P and

* CROFTON, *Proc. London Mathematical Society*, III. Demonstrations not substantially different from this will be found in GUDERMANN'S *Lehrbuch der niederen Sphärik*, § 144, and in a paper by ESSEN in GRUNERT'S *Archiv der Mathematik*, XXVII, 1856.

Q are right angles, the triangles AVP, BVQ are equal in all respects.

Hence it will be seen that

$$\left.\begin{aligned} V\hat{A}P = V\hat{B}Q = \tfrac{1}{2}(A - B), \\ V\hat{A}M = V\hat{B}M = \tfrac{1}{2}(A + B); \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (51)$$

also, since $CP = CQ$,

$$BQ = AP = \tfrac{1}{2}(a + b), \qquad CQ = CP = \tfrac{1}{2}(a - b). \quad \ldots\ldots (52)$$

Now $A\hat{V}P = B\hat{V}Q$; adding $A\hat{V}Q$ to both, $Q\hat{V}P = B\hat{V}A$, these two angles being bisected by VC and VM. Hence, applying the formula of Art. 54 to the right-angled triangles VMA, VPC,

$$\cos AM \sin V\hat{A}M = \cos M\hat{V}A = \cos C\hat{V}P = \cos CP \sin V\hat{C}P,$$

that is, $\cos\tfrac{1}{2}c \sin\tfrac{1}{2}(A + B) = \cos\tfrac{1}{2}C \cos\tfrac{1}{2}(a - b)$, ... (53)

the first of DELAMBRE'S formulae.

Again, applying the formulae of Art. 49 to the right-angled triangles AVP, CVP, we get

$$\sin\tfrac{1}{2}(a + b) \cot VP = \cot\tfrac{1}{2}(A - B), \quad \ldots\ldots\ldots\ldots (54)$$

$$\sin\tfrac{1}{2}(a - b) \cot VP = \cot\tfrac{1}{2}(\pi - C); \quad \ldots\ldots\ldots\ldots (55)$$

dividing, $$\frac{\sin\tfrac{1}{2}(a - b)}{\sin\tfrac{1}{2}(a + b)} = \tan\tfrac{1}{2}C \tan\tfrac{1}{2}(A - B), \quad \ldots\ldots\ldots\ldots (56)$$

and this is the second of NAPIER'S analogies.

The remaining formulae can be derived from the same construction, or an analogous one in which the angle C is bisected *internally*.

In connection with DELAMBRE'S analogies, (43)-(46), it is worthy of remark that the first and fourth are derivable from one another by the method of Art. 28. The second and third are unaltered by substitution of the elements of the polar triangle; they are, however, derivable from one another by substitution of the elements of a colunar triangle.

66. The formulae in the present chapter may be applied to establish analytically various propositions respecting spherical triangles which either have been proved geometrically in the preceding chapter, or may be so proved. Thus, for example, the second of NAPIER'S analogies is

$$\tan \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \tfrac{1}{2}C;$$

this shews that $\frac{1}{2}(A - B)$ is positive, negative, or zero, according as $\frac{1}{2}(a - b)$ is positive, negative, or zero; thus we obtain all the results included in Arts. 35...38.

67. *If two spherical triangles have two sides of the one equal to two sides of the other, each to each, but the angle which is contained by the two sides of the one greater than the angle which is contained by the two sides of the other which are equal to them, the base of that which has the greater angle will be greater than the base of the other; and conversely.*

Let b and c denote the sides which are equal in the two triangles; let a be the base and A the opposite angle of one triangle, and a' and A' similar quantities for the other. Then

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

$$\cos a' = \cos b \cos c + \sin b \sin c \cos A';$$

therefore $\quad \cos a - \cos a' = \sin b \sin c (\cos A - \cos A');$

that is,

$$\sin \tfrac{1}{2}(a + a') \sin \tfrac{1}{2}(a - a') = \sin b \sin c \sin \tfrac{1}{2}(A + A') \sin \tfrac{1}{2}(A - A');$$

this shews that $\frac{1}{2}(a - a')$ and $\frac{1}{2}(A - A')$ are of the same sign.

68. *If any point, other than the pole, be taken within a circle on the sphere, of all the arcs which can be drawn from that point to the circumference the greatest is that which passes through the pole, and the least that whose production passes through the pole; and of any others, that which is nearer to the greatest is always greater than one more remote; and from the same point to the circumference there*

can be drawn only two arcs which are equal to each other, and these make equal angles with the shortest arc, on opposite sides of it.

This follows readily from the preceding Article.

69. Reidt's analogies.

From DELAMBRE'S analogies we can deduce four others which may be used in the solution of triangles; they are given by Dr. FRIEDRICH REIDT in his *Sammlung von Ausgaben aus der Trigonometrie und Stereometrie*, 1872.*

In demonstrating them it is convenient to make use of the following abbreviated notation :

$$\left.\begin{array}{lll} \mathrm{A}+a=4s, & \mathrm{B}+b=4s', & \mathrm{C}+c=4s'', \\ \mathrm{A}-a=4d, & \mathrm{B}-b=4d', & \mathrm{C}-c=4d''. \end{array}\right\} \quad \ldots\ldots\ldots(57)$$

From DELAMBRE'S second analogy, Art. 63, (44), we obtain

$$\frac{\cos\frac{1}{2}\mathrm{C}-\sin\frac{1}{2}c}{\cos\frac{1}{2}\mathrm{C}+\sin\frac{1}{2}c}=\frac{\sin\frac{1}{2}(\mathrm{A}-\mathrm{B})-\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(\mathrm{A}-\mathrm{B})+\sin\frac{1}{2}(a-b)}; \quad \ldots.\ (58)$$

and, when in this we substitute $\sin(90°-\frac{1}{2}\mathrm{C})$ for $\cos\frac{1}{2}\mathrm{C}$, it readily reduces to

$$\tan(45°-s'')\cot(45°-d'')=\cot(s-s')\tan(d-d'). \quad \ldots(59)$$

In like manner DELAMBRE'S third analogy is equivalent to

$$\tan(45°-s'')\tan(45°-d'')=\tan(s+s')\tan(d+d'), \quad \ldots(60)$$

while similar treatment reduces the first and fourth to

$$\tan d''\tan s''=\tan(45°-s-d')\tan(45°-d-s'). \quad \ldots\ (61)$$

and
$$\tan d''\cot s''=\tan(45°-s+d')\cot(45°-d+s'). \quad \ldots\ (62)$$

From these we obtain other formulae by multiplication and division. Thus multiplication of the corresponding sides of (59) and (60) gives

$$\tan^2(45°-s'')=\cot(s-s')\tan(s+s')\tan(d-d')\tan(d+d'), \quad (63)$$

and division gives

$$\tan^2(45°-d'')=\tan(s-s')\tan(s+s')\cot(d-d')\tan(d+d'). \quad (64)$$

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* Fourth Edition, Vol. I, § 37, p. 232. Cf. SERRET, *Traité de Trigonométrie*, § 123.

Similarly from (61) and (62)

$$\tan^2 d'' = \tan(45^\circ - s - d')\tan(45^\circ - s + d')\tan(45^\circ - d - s')$$
$$\tan(45^\circ + d - s'), \quad \ldots\ldots (65)$$

$$\tan^2 s'' = \tan(45^\circ - s - d')\tan(45^\circ + s - d')\tan(45^\circ - d - s')$$
$$\tan(45^\circ - d + s'). \quad \ldots\ldots (66)$$

These last four relations are REIDT'S analogies.

In connection with the ambiguous case in the solution of spherical triangles (see Art. 109) it is important to notice the result of applying the formulae just obtained to a second triangle whose elements a, b, and A are the same as those of the original triangle, but whose angle opposite to the side b is $180^\circ - B$. If C′ and c' be the third angle and the third side, formulae (65) and (66) give:

$$\tan^2\tfrac{1}{4}(C' - c') = \tan(s' - s)\cot(s' + s)\tan(d' - d)\tan(d' + d), \quad (67)$$

$$\tan^2\tfrac{1}{4}(C' + c) = \tan(s' - s)\tan(s' + s)\tan(d' - d)\cot(d' + d). \quad (68)$$

When a, b, and B are given, and A has been found, formulae (65) and (66) may be used to determine C and c, formulae (67) and (68) to determine C′ and c'.

70. We shall give another proof of the fundamental formulae of Art. 44, which is very simple, requiring only a knowledge of the elements of Coordinate Geometry.

Suppose ABC any spherical triangle, O the centre of the sphere, take O as the origin of coordinates, and let the axis of z pass through C. Let x_1, y_1, z_1 be the coordinates of A, and x_2, y_2, z_2 those of B; let r be the radius of the sphere. Then the square on the straight line AB is equal to

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2, \quad \ldots\ldots (69)$$

and also to

$$r^2 + r^2 - 2r^2 \cos A\hat{O}B; \quad \ldots\ldots (70)$$

and

$$x_1^2 + y_1^2 + z_1^2 = r^2, \quad x_2^2 + y_2^2 + z_2^2 = r^2,$$

thus

$$x_1x_2 + y_1y_2 + z_1z_2 = r^2 \cos A\hat{O}B. \quad \ldots\ldots (71)$$

Now make the usual substitutions in passing from rectangular to polar co-ordinates, namely,

$$\left.\begin{array}{lll} z_1 = r\cos\theta_1, & x_1 = r\sin\theta_1\cos\phi_1, & y_1 = r\sin\theta_1\sin\phi_1, \\ z_2 = r\cos\theta_2, & x_2 = r\sin\theta_2\cos\phi_2, & y_2 = r\sin\theta_2\sin\phi_2; \end{array}\right\} \quad (72)$$

thus we obtain

$$\cos\theta_2\cos\theta_1 + \sin\theta_2\sin\theta_1\cos(\phi_1 - \phi_2) = \cos \mathsf{A\hat{O}B}, \ \ldots (73)$$

that is, in the ordinary notation of Spherical Trigonometry,

$$\cos a\cos b + \sin a\sin b\cos \mathsf{C} = \cos c. \ \ldots\ldots\ldots\ldots (74)$$

This method, like that of Art. 52, has the advantage of giving a *perfectly general proof*, as all the equations used are *universally* true.

EXAMPLES I.

1. If $\mathsf{A} \mp a$, shew that B and b are equal or supplemental, as also C and c.

2. If one angle of a triangle be equal to the sum of the other two, the greatest side is double of the distance of its middle point from the opposite angle.

3. When does the polar triangle coincide with the primitive triangle?

4. If D be the middle point of AB, shew that

$$\cos \mathsf{AC} + \cos \mathsf{BC} = 2\cos\tfrac{1}{2}\mathsf{AB}\cos \mathsf{CD}.$$

5. If two angles of a spherical triangle be respectively equal to the sides opposite to them, shew that the remaining side is the supplement of the remaining angle; or else that the triangle has two quadrants and two right angles, and then the remaining side is equal to the remaining angle.

6. In an equilateral triangle, shew that $2\cos\frac{1}{2}a\sin\frac{1}{2}\mathsf{A} = 1$.

7. In an equilateral triangle, shew that $\tan^2\frac{1}{2}a = 1 - 2\cos\mathsf{A}$; hence deduce the limits between which the sides and the angles of an equilateral triangle are restricted.

8. In an equilateral triangle, shew that $\sec\mathsf{A} = 1 + \sec a$.

9. If the three sides of a spherical triangle be halved and a new triangle formed, the angle θ between the new sides $\frac{1}{2}b$ and $\frac{1}{2}c$ is given by $\cos\theta = \cos\mathsf{A} + \frac{1}{2}\tan\frac{1}{2}b\tan\frac{1}{2}c\sin^2\theta$.

10. AB, CD are quadrants on the surface of a sphere intersecting at E, the extremities being joined by great circles: shew that

$$\cos A\hat{E}C = \cos AC \cos BD - \cos BC \cos AD.$$

11. If $b + c = \pi$, shew that $\sin 2B + \sin 2C = 0$.

12. If DE be an arc of a great circle bisecting the sides AB, AC of a spherical triangle at D and E, P a pole of DE, and PB, PD, PE, PC be joined by arcs of great circles, shew that the angle BPC = twice the angle DPE.

13. In a spherical triangle shew that

$$\sin b \sin c + \cos b \cos c \cos A = \sin B \sin C - \cos B \cos C \cos a.$$

(CAGNOLI).

14. If D be any point in the side BC of a triangle, shew that

$$\cos AD \sin BC = \cos AB \sin DC + \cos AC \sin BD.$$ (Cf. Art. 143.)

15. In a spherical triangle shew that if θ, ϕ, ψ be the arcs of great circles drawn from A, B, C perpendicular to the opposite sides,

$$\sin a \sin \theta = \sin b \sin \phi = \sin c \sin \psi$$
$$= \sqrt{(1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c)}.$$

16. In a spherical triangle, if θ, ϕ, ψ be the arcs bisecting the angles A, B, C respectively and terminated by the opposite sides, shew that

$$\cot \theta \cos \tfrac{1}{2}A + \cot \phi \cos \tfrac{1}{2}B + \cot \psi \cos \tfrac{1}{2}C = \cot a + \cot b + \cot c.$$

17. Two ports are in the same parallel of latitude, their common latitude being l and their difference of longitude 2λ: shew that the saving of distance in sailing from one to the other on the great circle, instead of sailing due East or West, is

$$2r\{\lambda \cos l - \sin^{-1}(\sin \lambda \cos l)\},$$

λ being expressed in circular measure, and r being the radius of the Earth.

18. If a ship be proceeding uniformly along a great circle and the observed latitudes be l_1, l_2, l_3 at equal intervals of time, in each of which the distance traversed is s, shew that

$$s = r \cos^{-1} \frac{\sin \frac{1}{2}(l_1 + l_3) \cos \frac{1}{2}(l_1 - l_3)}{\sin l_2},$$

r denoting the Earth's radius: and shew that the change of longitude may also be found in terms of the three latitudes.

EXAMPLES II.

1. In any spherical triangle shew that

$$2\cos\tfrac{1}{2}(a+b)\cos\tfrac{1}{2}(a-b)\tan\tfrac{1}{2}c=\sin b\cos \mathrm{A}+\sin a\cos \mathrm{B};$$

and that

$$\tan\tfrac{1}{2}(\mathrm{A}-a)\tan\tfrac{1}{2}(\mathrm{B}+b)=\tan\tfrac{1}{2}(\mathrm{B}-b)\tan\tfrac{1}{2}(\mathrm{A}+a).$$

2. Prove that

$$\cos a\tan \mathrm{B}+\cos b\tan \mathrm{A}+\tan \mathrm{C}=\cos a\cos b\tan \mathrm{A}\tan \mathrm{B}\tan \mathrm{C}.$$

3. Given two meridians of a sphere, through a given point on a meridian bisecting the angle between them is drawn a variable great circle cutting them in the points P and Q; prove that the sum of the tangents of the latitudes of P and Q is constant. (R. U. I., 1898.)

4. In a spherical triangle whose sides are each less than 90°, prove that an exterior angle is greater than either of the interior and opposite angles. (R. U. I., 1893.)

5. If P is taken in AB, a side of any triangle ABC, such that AP equals AC, shew that

$$\sin c\cos \mathrm{CP}=\cos a\sin b+\cos b\sin(c-b).$$

(Sci. and Art, 1895.)

6. Shew that

$$\frac{\sin c}{\sin \mathrm{C}}=\sqrt{\frac{1-\cos a\cos b\cos c}{1+\cos \mathrm{A}\cos \mathrm{B}\cos \mathrm{C}}}.$$

(Sci. and Art, 1897.)

7. The sides AB, BC of a spherical quadrilateral ABCD are denoted by a, b respectively, and the angle ABD by θ; shew that

$$\tan\theta=\frac{\cos a\sin b-\sin a(\cos b\cos \mathrm{B}+\cot \mathrm{C}\sin \mathrm{B})}{-\cot \mathrm{A}\sin b+\sin a(\cos b\sin \mathrm{B}-\cot \mathrm{C}\cos \mathrm{B})}.$$

(Sci. and Art, 1898.)

8. If corresponding angles of a triangle ABC and its polar triangle are equal, shew that

$$\sec^2 \mathrm{A}+\sec^2 \mathrm{B}+\sec^2 \mathrm{C}+2\sec \mathrm{A}\sec \mathrm{B}\sec \mathrm{C}=1.$$

(Sci. and Art, 1898.)

9. If $\mathrm{A}=a$, shew that

$$\tan\tfrac{1}{2}a=\frac{\tan\tfrac{1}{2}b \sim \tan\tfrac{1}{2}c}{1-\tan\tfrac{1}{2}b\tan\tfrac{1}{2}c}.$$

(Sci. and Art, 1899.)

10. If the sum of two angles of a spherical triangle is less than π, shew that the sum of the opposite sides is less than the semi-circumference of a great circle.

11. Prove that in any triangle

$$\frac{\sin(A+B)}{\sin C}=\frac{\cos a+\cos b}{1+\cos c}.$$

[MORGAN JENKINS (*Messenger of Mathematics*, XVII, p. 30) regards this as a fundamental formula of the triangle, and deduces from it the analogies of NAPIER and DELAMBRE.]

12. L, L', L'', L''' are four points on a sphere, and A is the angle between the arcs LL' and $L''L'''$. Shew that

$$\cos LL''\cos L'L''' - \cos LL'''\cos L'L'' = \sin LL'\sin L''L'''\cos A.$$

(GAUSS.)

CHAPTER IV.

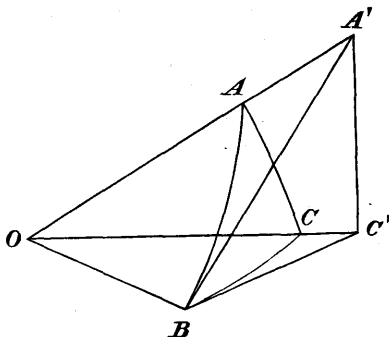
SOLUTION OF RIGHT-ANGLED TRIANGLES.

71. In every spherical triangle there are six elements, namely the three sides and the three angles, besides the radius of the sphere, which is supposed to be a known constant. The solution of spherical triangles is the process by which, when the values of a sufficient number of the six elements are given, we calculate the values of the remaining elements. It will appear, as we proceed, that when the values of three of the elements are given, those of the remaining three can generally be found. We begin with the right-angled triangle: here two elements, in addition to the right angle, will be supposed known.

72. The formulae requisite for the solution of right-angled triangles may be obtained as particular cases of the formulae of the preceding chapter, by supposing one of the angles a right angle, as C for example. They may also be obtained very easily in an independent manner, as we shall now shew.

73. Formulae of the right-angled triangle. Let ABC be a spherical triangle having a right angle at C; let O be the centre of the sphere.

Draw the tangent at B to the great circle BC; this tangent lies in the plane BOC of the great circle, and will therefore meet OC produced; let the point of intersection be denoted

by C′. Also draw the tangent at B to the great circle BA; this will, for the same reason, intersect OA produced, and the point of intersection may be called A′. Join A′C′.

Since the radius OB is perpendicular to both the tangents BC′ and BA′, it follows that the plane A′BC′ is perpendicular to every plane through OB, and therefore to the plane BOC. And, by hypothesis, the plane AOC is perpendicular to the plane BOC. Hence the line A′C′, being the intersection of the planes AOC and A′BC′, is perpendicular to the plane BOC. Therefore the angles OC′A and BC′A are right. Thus the diagram contains four right-angled triangles A′BC′, A′OC′, C′OB, and A′OB, having the angles A′BC′, A′OC′, C′OB, and A′OB equal respectively to the elements B, b, a, and c of the spherical triangle.

Now $$\frac{OB}{OA'} = \frac{OB}{OC'} \cdot \frac{OC'}{OA'},$$

and therefore $$\cos c = \cos a \cos b. \quad \text{......................(1)}$$

Again, $$\sin B = \frac{C'A'}{BA'} = \frac{C'A'}{OA'} \cdot \frac{OA'}{BA'} = \frac{\sin b}{\sin c},$$

that is, similarly $$\left.\begin{aligned} \sin b &= \sin B \sin c\,; \\ \sin a &= \sin A \sin c. \end{aligned}\right\} \quad \text{......................(2)}$$

$$\text{Cos B} = \frac{\text{BC}'}{\text{BA}'} = \frac{\text{BC}'}{\text{OB}} \cdot \frac{\text{OB}}{\text{BA}'} = \frac{\tan a}{\tan c},$$

that is, similarly
$$\left.\begin{aligned} \tan a &= \cos \text{B} \tan c; \\ \tan b &= \cos \text{A} \tan c. \end{aligned}\right\} \quad \ldots\ldots(3)$$

$$\text{Tan B} = \frac{\text{C}'\text{A}'}{\text{BC}'} = \frac{\text{C}'\text{A}'}{\text{OC}'} \cdot \frac{\text{OC}'}{\text{BC}'} = \frac{\tan b}{\sin a},$$

that is, similarly
$$\left.\begin{aligned} \tan b &= \tan \text{B} \sin a; \\ \tan a &= \tan \text{A} \sin b. \end{aligned}\right\} \quad \ldots\ldots(4)$$

Multiply together the two formulae (4); thus,

$$\tan \text{A} \tan \text{B} = \frac{\tan a \tan b}{\sin a \sin b} = \frac{1}{\cos a \cos b} = \frac{1}{\cos c}, \text{ by (1)};$$

therefore
$$\cos c = \cot \text{A} \cot \text{B}. \quad \ldots\ldots(5)$$

Multiply crosswise the second formula in (2) and the first in (3); thus, $\sin a \cos \text{B} \tan c = \tan a \sin \text{A} \sin c$;

therefore
$$\cos \text{B} = \frac{\sin \text{A} \cos c}{\cos a} = \sin \text{A} \cos b, \text{ by (1)}.$$

Thus similarly
$$\left.\begin{aligned} \cos \text{B} &= \sin \text{A} \cos b; \\ \cos \text{A} &= \sin \text{B} \cos a. \end{aligned}\right\} \quad \ldots\ldots(6)$$

These six formulae comprise ten equations; and thus we can solve every case of right-angled triangles. For every one of these ten equations is a distinct combination involving three out of the five quantities a, b, c, A, B; and out of five quantities only ten combinations of three can be formed. Thus any two of the five quantities being given and a third required, some one of the preceding ten equations will serve to determine that third quantity.

A three-dimensional model* of the diagram of this Article may be made simply as follows. On a piece of stiff paper describe a circle with centre O and radius equal to that of the sphere. Mark on the circumference points B, C, A, B_1, such that $\text{B}\hat{\text{O}}\text{C}$, $\text{C}\hat{\text{O}}\text{A}$, and $\text{A}\hat{\text{O}}\text{B}_1$ are equal to the elements a, b, c of the spherical triangle. Draw tangents BC′, $\text{B}_1\text{A}'$

* Suggested by Prof. G. H. Bryan.

meeting OC, OA in C′, A′ respectively. Now cut the paper along the lines OB, BC′, C′A′, A′B_1, B_1O, make creases along OC′ and OA′, and bend up till OB and OB_1 coincide.

74. As we have stated, the above six formulae may be obtained from those given in the preceding chapter by supposing C a right angle. Thus, (1) follows from Art. 44, (2) from Art. 46, (3) from the fourth and fifth equations of Art. 49, (4) from the first and second equations of Art. 49, (5) from the third equation of Art. 54, (6) from the first and second equations of Art. 54.

Since the six formulae may be obtained from those given in the preceding chapter which have been proved to be universally true, we do not stop to shew that the demonstration of Art. 73 may be applied to every case which can occur; the student may for exercise investigate the modifications which will be necessary when we suppose one or more of the quantities a, b, c, A, B equal to a right angle or greater than a right angle.

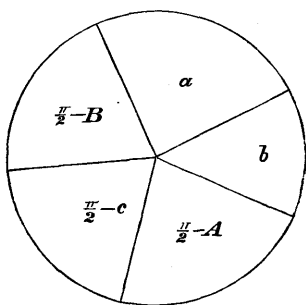
75. Certain properties of right-angled triangles are deducible from the formulae of Art. 73.

From (1) it follows that $\cos c$ has the same sign as the product $\cos a \cos b$; hence either all the cosines are positive, or else only one is positive. Therefore *in a right-angled triangle either all the three sides are less than quadrants, or else one side is less than a quadrant and the other two sides are greater than quadrants.*

From (4) it follows that $\tan a$ has the same sign as $\tan A$. Therefore A and a are either both greater than $\frac{1}{2}\pi$, or both less than $\frac{1}{2}\pi$; this is expressed by saying that A and a are *of the same affection.* Similarly B and b are of the same affection.

76. Napier's Rules. The formulae of Art. 73 are comprised in two rules, which are called, from their inventor, NAPIER'S

Rules of Circular Parts. NAPIER was also the inventor of Logarithms, and the Rules of Circular Parts were first published by him in a work entitled *Mirifici Logarithmorum Canonis Descriptio*......Edinburgh, 1614. These rules we shall now explain.

The right angle is left out of consideration; the two sides which include the right angle, the complement of the hypotenuse, and the complements of the other angles are called the *circular parts* of the triangle. Thus there are *five* circular parts, namely, a, b, $\frac{1}{2}\pi - \mathrm{A}$, $\frac{1}{2}\pi - c$, $\frac{1}{2}\pi - \mathrm{B}$; and these are supposed to be ranged round a circle in the order in which they naturally occur with respect to the triangle.

Any one of the five parts may be selected and called the *middle part*, then the two parts next to it are called *adjacent parts*, and the remaining two parts are called *opposite parts.* For example, if $\frac{1}{2}\pi - \mathrm{B}$ is selected as the middle part, then the adjacent parts are a and $\frac{1}{2}\pi - c$, and the opposite parts are b and $\frac{1}{2}\pi - \mathrm{A}$.

Then NAPIER'S Rules are the following:

sine of the middle part = product of tangents of adjacent parts;

sine of the middle part = product of cosines of opposite parts.

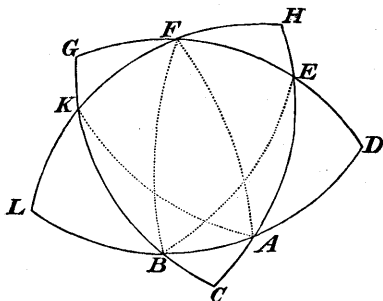
77. NAPIER's Rules may be demonstrated by shewing that they agree with the results already established. The following table shews the required agreement: in the first column are given the *middle parts*, in the second column the results of NAPIER's Rules, and in the third column the same results expressed as in Art. 73, with the number for reference used in that Article.

$\frac{1}{2}\pi - c$	$\sin(\frac{1}{2}\pi - c) = \tan(\frac{1}{2}\pi - A)\tan(\frac{1}{2}\pi - B)$ $\sin(\frac{1}{2}\pi - c) = \cos a \cos b$	$\cos c = \cot A \cot B$....(5) $\cos c = \cos a \cos b$. ...(1)
$\frac{1}{2}\pi - B$	$\sin(\frac{1}{2}\pi - B) = \tan a \tan(\frac{1}{2}\pi - c)$ $\sin(\frac{1}{2}\pi - B) = \cos b \cos(\frac{1}{2}\pi - A)$	$\cos B = \tan a \cot c$....(3) $\cos B = \cos b \sin A$. .. (6)
a	$\sin a = \tan b \tan(\frac{1}{2}\pi - B)$ $\sin a = \cos(\frac{1}{2}\pi - A)\cos(\frac{1}{2}\pi - c)$	$\sin a = \tan b \cot B$....(4) $\sin a = \sin A \sin c$. ...(2)
b	$\sin b = \tan(\frac{1}{2}\pi - A)\tan a$ $\sin b = \cos(\frac{1}{2}\pi - B)\cos(\frac{1}{2}\pi - c)$	$\sin b = \cot A \tan a$....(4) $\sin b = \sin B \sin c$. ...(2)
$\frac{1}{2}\pi - A$	$\sin(\frac{1}{2}\pi - A) = \tan b \tan(\frac{1}{2}\pi - c)$ $\sin(\frac{1}{2}\pi - A) = \cos a \cos(\frac{1}{2}\pi - B)$	$\cos A = \tan b \cot c$. ...(3) $\cos A = \cos a \sin B$....(6)

The last four cases need not have been given, since it is obvious that they are only repetitions of what had previously been given; the seventh and eighth are repetitions of the fifth and sixth, and the ninth and tenth are repetitions of the third and fourth.

78. It has sometimes been stated that the method of the preceding Article is the only one by which NAPIER's Rules can be demonstrated; this statement, however, is inaccurate, since besides this method NAPIER himself indicated another method of proof in his *Mirifici Logarithmorum Canonis Descriptio*, pp. 32, 35. This we shall now briefly explain.

Let ABC be a spherical triangle right-angled at C; with B as pole describe a great circle DEFG, and with A as pole describe a great circle HFKL, and produce the sides of the original triangle ABC to meet these great circles. Then since B is a pole of DEFG the angles at D and G are right angles, and since A is a pole of HFKL the angles at H and L are right angles. Hence the five triangles BAC, AED, EFH, FKG, KBL are all *right-angled*; and moreover it will be found on examination that, although the elements of these triangles are

different, yet *their circular parts are the same.* We will consider, for example, the triangle AED; the angle EAD is equal to the angle BAC; the side AD is the complement of AB; as the angles at C and G are right angles E is a pole of GC (Art. 13), therefore EA is the complement of AC; as B is a pole of DE the angle BED is a right angle, therefore the angle AED is the complement of the angle BEC, that is, the angle AED is the complement of the side BC (Art. 12); and similarly the side DE is equal to the angle DBE, and is therefore the complement of the angle ABC. Hence, if we denote the elements of the triangle ABC as usual by a, b, c, A, B, we have in the triangle AED the hypotenuse equal to $\frac{1}{2}\pi - b$, the angles equal to A and $\frac{1}{2}\pi - a$, and the sides respectively opposite these angles equal to $\frac{1}{2}\pi$ - B and $\frac{1}{2}\pi - c$. The *circular parts* of AED are therefore the same as those of ABC. Similarly the remaining three of the five right-angled triangles may be shewn to have the same circular parts as the triangle ABC has.

Now take *two* of the theorems in Art. 73, for example (1) and (3): then the truth of the *ten* cases comprised in NAPIER's Rules will be found to follow from applying the two theorems in succession to the five triangles formed in the preceding figure. Thus this method of considering NAPIER's Rules regards each Rule, not as the statement of dissimilar properties of one triangle, but as the statement of similar properties of five allied triangles.

79. In NAPIER's work a figure is given of which that in the preceding Article is a copy, except that different letters are used; NAPIER briefly intimates that the truth of the Rules can easily be seen by means of this figure, as well as by the method of induction from consideration

of all the cases which can occur. The late T. S. DAVIES, in his edition of Dr. HUTTON'S *Course of Mathematics*, drew attention to NAPIER'S own views and expanded the demonstration by a systematic examination of the figure of the preceding Article.

It is however easy to evade the necessity of examining the whole figure; all that is wanted is to observe the connexion between the triangle AED and the triangle BAC. For let a_1, a_2, a_3, a_4, a_5 represent the elements of the triangle BAC taken in order, beginning with the hypotenuse and omitting the right angle; then the elements of the triangle AED taken in order, beginning with the hypotenuse and omitting the right angle, are $\frac{1}{2}\pi - a_3$, $\frac{1}{2}\pi - a_4$, $\frac{1}{2}\pi - a_5$, $\frac{1}{2}\pi - a_1$ and a_2. If, therefore, to characterise the former we introduce a new set of quantities p_1, p_2, p_3, p_4, p_5, such that $a_1 + p_1 = a_2 + p_2 = a_5 + p_5 = \frac{1}{2}\pi$, and that $p_3 = a_3$ and $p_4 = a_4$, then the original triangle being characterised by p_1, p_2, p_3, p_4, p_5, the second triangle will be similarly characterised by p_3, p_4, p_5, p_1, p_2. As the second triangle can give rise to a third in like manner, and so on, we see that every right-angled triangle is one of a system of five such triangles which are all characterised by the quantities p_1, p_2, p_3, p_4, p_5, always taken in order, each quantity in its turn standing first.

The late R. L. ELLIS pointed out this connexion between the five triangles, and thus gave the true significance of NAPIER'S Rules. The memoir containing Mr. ELLIS'S investigations, which was unpublished when the first edition of the present work appeared, will be found in pages 328-335 of *The Mathematical and other writings of Robert Leslie Ellis* ... Cambridge, 1863.

NAPIER'S own method of considering his Rules was neglected by writers on the subject until the late T. S. DAVIES drew attention to it. Hence, as we have already remarked in Art. 78, an erroneous statement was made respecting the Rules. For instance, WOODHOUSE says, in his *Trigonometry*; "There is no separate and independent proof of these rules; ..." AIRY says, in the treatise on Trigonometry in the *Encyclopædia Metropolitana*: "These rules are proved to be true only by showing that they comprehend all the equations which we have just found." *

* A discussion of NAPIER'S Rules of Circular Parts by Prof. E. O. LOVETT will be found in the *Bulletin of the American Mathematical Society*, Second Series, IV, 1898, p. 252. See also O. PUND, *Ueber Substitutionsgruppen in der sphärischen Trigonometrie, Mitteilungen der math. Gesell. in Hamburg*, III, 1897-8, p. 290.

80. Opinions have differed with respect to the *utility* of NAPIER'S Rules in practice. Thus WOODHOUSE says, "In the whole compass of mathematical science there cannot be found, perhaps, rules which more completely attain that which is the proper object of rules, namely, facility and brevity of computation" (*Trigonometry*, Chap. X.). On the other hand may be set the following sentence from AIRY'S Trigonometry (*Encyclopædia Metropolitana*): "In the opinion of DELAMBRE (and no one was better qualified by experience to give an opinion) these theorems are best recollected by the practical calculator in their unconnected form." See DELAMBRE'S *Astronomie*, Vol. I, p. 205. Professor DE MORGAN strongly objects to NAPIER'S Rules, and says (*Spherical Trigonometry*, Art. 17): "There are certain mnemonical formulae called NAPIER'S *Rules of Circular Parts*, which are generally explained. We do not give them, because we are convinced that they only create confusion instead of assisting the memory."

81. Solution of right-angled triangles. We shall now proceed to apply the formulae of Art. 73 to the solution of right-angled triangles. We shall assume that the given quantities are subject to the limitations which are stated in Arts. 22 and 23, that is, a given side must be less than the semicircumference of a great circle, and a given angle less than two right angles.

In making numerical calculations from the formulae, use is made of logarithms. The student who has had practice in the solution of plane triangles will be familiar with the use of logarithmic and trigonometrical tables; others are referred to the author's book on *Plane Trigonometry*, where a chapter is devoted to the subject. There is just one caution which it seems desirable to repeat here; namely that the calculation of very small angles by their cosines, or of angles near 90° by their sines, is to be avoided, as the corresponding tabular logarithms vary very slowly. It is preferable to calculate such angles by their tangents.

There are six cases to be considered.

82. Case I.—*Having given the hypotenuse* c *and an angle* A. Here we have from (3), (5), and (2) of Art. 73,

$$\left.\begin{aligned}\tan b &= \tan c \cos A,\\ \cot B &= \cos c \tan A,\\ \sin a &= \sin c \sin A.\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Thus b and B are determined immediately without ambiguity; and as a must be of the same affection as A (Art. 75), a also is determined without ambiguity.

It is obvious, from the formulae of solution, that in this case the triangle is always possible.

If c and A are both right angles, a is a right angle, and b and B are indeterminate.

If a prove to be very near to 90°, which is the case when A and c are both very near to 90°, one must commence by calculating the values of b and B. Then a may be determined by either of the formulae

$$\tan a = \sin b \tan A, \quad \tan a = \tan c \cos B.$$

83. Case II.—*Having given a side* b *and the adjacent angle* A.

Here we have from (3), (4), and (6) of Art. 73,

$$\left.\begin{aligned}\tan c &= \frac{\tan b}{\cos A},\\ \tan a &= \tan A \sin b,\\ \cos B &= \cos b \sin A.\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Thus c, a, B are determined without ambiguity, and the triangle is always possible.

If B be small, which happens when A is very near to 90° and b is very near to 0° or 180°, a is first determined, and then use is made of the formula

$$\tan B = \frac{\tan b}{\sin a}.$$

84. Case III.—*Having given the two sides* a *and* b.

Here we have from (1) and (4) of Art. 73,

$$\left.\begin{aligned}\cos c &= \cos a \cos b,\\ \cot A &= \cot a \sin b,\\ \cot B &= \cot b \sin a.\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Thus c, A, B are determined without ambiguity, and the triangle is always possible.

If c be very small, which is the case when both a and b are very near to 0° or 180°, A and B are first determined, and then use is made of either of the formulae

$$\tan c = \frac{\tan a}{\cos B}, \quad \tan c = \frac{\tan b}{\cos A}.$$

85. Case IV.—*Having given the hypotenuse* c *and a side* a.

Here we have from (1), (3), and (2) of Art. 73,

$$\left.\begin{aligned} \cos b &= \frac{\cos c}{\cos a}, \\ \cos B &= \frac{\tan a}{\tan c}, \\ \sin A &= \frac{\sin a}{\sin c}. \end{aligned}\right\} \quad \text{..........................(10)}$$

Here b, B, A are determined without ambiguity, since A must be of the same affection as a. It will be seen from these formulae that a certain limitation of the data is requisite in order to insure a possible triangle; in fact, c must lie between a and $\pi - a$ in order that the values found for $\cos b$, $\cos B$, and $\sin A$ may be numerically not greater than unity.

If c and a are right angles, A is a right angle, and b and B are indeterminate.

86. In order to solve the triangle completely by the three equations given above, it is necessary to look out six logarithms. If, however, we use other formulae immediately deducible from them, namely,

$$\left.\begin{aligned} \tan \tfrac{1}{2}b &= +\sqrt{\tan \tfrac{1}{2}(c+a) \tan \tfrac{1}{2}(c-a)}, \\ \tan \tfrac{1}{2}B &= +\sqrt{\frac{\sin (c-a)}{\sin (c+a)}}, \\ \tan (45^\circ + \tfrac{1}{2}A) &= \pm\sqrt{\frac{\tan \tfrac{1}{2}(c+a)}{\tan \tfrac{1}{2}(c-a)}}, \end{aligned}\right\} \quad \text{....................(11)}$$

only four logarithms need be looked out. The latter forms are, moreover, those to be used when b or B is very small, or when A is near 90°. The signs prefixed to the square roots in the first two formulae are positive because $\frac{1}{2}b$ and $\frac{1}{2}$B cannot exceed 90. In the third the sign is positive or negative according as a is less or greater than 90°; for if a is less than 90°, so is A, and therefore also $45° + \frac{1}{2}A$.

87. Case V.—*Having given the two angles* A *and* B.

Here we have from (5) and (6) of Art. 73,

$$\left.\begin{aligned}\cos c &= \cot A \cot B,\\ \cos a &= \frac{\cos A}{\sin B},\\ \cos b &= \frac{\cos B}{\sin A}.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(12)$$

Here c, a, b are determined without ambiguity. There are, however, limitations of the data, requisite in order to insure a possible triangle; for, as $\cos a$ and $\cos b$ must be less than unity, it is necessary that cos A should be less than sin B, and cos B than sin A, numerically. Of course if one of these conditions is satisfied, so is the other, since $\sin^2 B - \cos^2 A = \sin^2 A - \cos^2 B$. First suppose A less than 90°, then cos B is to be numerically less than $\cos(90° - A)$, and hence B must lie between $90° - A$ and $90° + A$; next suppose A greater than 90°, then cos B is to be numerically less than $\cos(A - 90°)$, and therefore B must lie between $A - 90°$ and $180° - (A - 90°)$, that is, between $A - 90°$ and $270° - A$.

When one of the three sides is found to be so small that it cannot be calculated accurately from its cosine, the following modifications of the above formulae (requiring only four logarithms) should be used:

$$\left.\begin{aligned}\tan \tfrac{1}{2}c &= +\sqrt{\frac{\sin(A+B-90°)}{\cos(A-B)}},\\ \tan \tfrac{1}{2}a &= +\sqrt{\tan \tfrac{1}{2}(A+B-90°)\cot \tfrac{1}{2}(B-A+90°)},\\ \tan \tfrac{1}{2}b &= +\sqrt{\tan \tfrac{1}{2}(A+B-90°)\tan \tfrac{1}{2}(B-A+90°)}.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(13)$$

88. Case VI.—*Having given a side* a *and the opposite angle* A. Here we have from (2), (4), and (6) of Art. 73,

$$\left.\begin{aligned} \sin c &= \frac{\sin a}{\sin A}, \\ \sin b &= \tan a \cot A, \\ \sin B &= \frac{\cos A}{\cos a}. \end{aligned}\right\} \quad \text{......................(14)}$$

Now there is only one angle between 0° and 180° that has a given cosine or a given tangent, but there are in general two angles having a given sine. Hence an element of a spherical triangle is determined *uniquely* when its tangent or its cosine is known, but *ambiguously* when only its sine is known. In the present instance all three of the sought parts have to be inferred from their sines, and so there is a treble ambiguity; from this it might be supposed that there are six different triangles satisfying the data, but a little consideration shews that there are only two. Of course $\sin a$ must be less than $\sin A$, or the value obtained for $\sin c$ would be inadmissible. When this requirement is complied with, there are two values admissible for c; corresponding to each of these there will be *in general* only one admissible value of b, since we must have $\cos c = \cos a \cos b$, an equation determining b by its cosine, and therefore uniquely; and likewise only one admissible value of B, since it is determined by its cotangent, and therefore uniquely, from the relation $\cos c = \cot A \cot B$.

Thus if one triangle exists with the given parts, there will be *in general* two, and only two, triangles with the given parts. We say *in general* in the preceding sentences, because if $a = A$ there will be only *one* triangle, unless a and A are each right angles, and then b and B become indeterminate.

It is easy to see from a figure that the ambiguity must occur in general.

For, suppose BAC to be a triangle which satisfies the given

conditions; produce AB and AC to meet again at A′; then the triangle A′BC also satisfies the given conditions, for it has a right angle at C, BC the given side, and $A' = A$ the given angle.

If $a = A$, then the formulae of solution shew that c, b, and B are right angles; in this case A is the pole of BC, and the triangle A′BC is symmetrically equal to the triangle ABC.

If a is a right angle, which involves that A also be a right angle, B is the pole of AC; B and b are then equal, but may have any value whatever.

Certain limitations of the data are requisite in order to insure a possible triangle. A and a must have the same affection by Art. 75; and, in order that the values of the sines given by the formulae of solution may be between -1 and $+1$, it is necessary that $\sin a$ be numerically less than $\sin A$; hence a must be less than A if both are acute, and greater than A if both are obtuse.

When any of the sought parts are such that they cannot be calculated with sufficient accuracy from their sines, the following modifications of the formulae of solution should be used:

$$\left.\begin{aligned} \tan(45° - \tfrac{1}{2}c) &= \pm\sqrt{\tan\tfrac{1}{2}(A - a)\cot\tfrac{1}{2}(A + a)}, \\ \tan(45° - \tfrac{1}{2}b) &= \pm\sqrt{\frac{\sin(A - a)}{\sin(A + a)}}, \\ \tan(45° - \tfrac{1}{2}B) &= \pm\sqrt{\tan\tfrac{1}{2}(A - a)\tan\tfrac{1}{2}(A + a)}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(15)$$

These supplemental formulae, and those of Case IV, are given by CAGNOLI.*

89. Application of Napier's analogies. Col. CLARKE, in his work on *Geodesy*, p. 43, points out that in three of the cases

* *Trigonometria*, §§ 1030-1035.

the solution of the right-angled triangle may be considerably shortened by making use of NAPIER'S analogies.

Suppose, in the first place, that the sides a and b are given. Write V for 90° − A, and we have from the first two analogies

$$\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)} = \tan\tfrac{1}{2}(\mathrm{A}+\mathrm{B}) = \frac{1+\tan\frac{1}{2}(\mathrm{B}-\mathrm{V})}{1-\tan\frac{1}{2}(\mathrm{B}-\mathrm{V})}, \quad \ldots\ldots (16)$$

$$\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)} = \tan\tfrac{1}{2}(\mathrm{A}-\mathrm{B}) = \frac{1-\tan\frac{1}{2}(\mathrm{B}+\mathrm{V})}{1+\tan\frac{1}{2}(\mathrm{B}+\mathrm{V})}; \quad \ldots\ldots (17)$$

whence the following,

$$\tan\tfrac{1}{2}(\mathrm{B}-\mathrm{V}) = \tan\tfrac{1}{2}a\,\tan\tfrac{1}{2}b, \quad \ldots\ldots\ldots\ldots (18)$$

$$\tan\tfrac{1}{2}(\mathrm{B}+\mathrm{V}) = \cot\tfrac{1}{2}a\,\tan\tfrac{1}{2}b; \quad \ldots\ldots\ldots\ldots (19)$$

when these formulae are used to find B and V, only two logarithms are looked out instead of four. From these also, if A and B be given, a and b are easily obtained, multiplication and division of the present formulae leading in fact to the formulae given under Case V.

Again if the side b and the adjacent angle A be given, we use the third and fourth of NAPIER'S analogies, which in this case take the form

$$\tan\tfrac{1}{2}(c+a) = \tan\tfrac{1}{2}b\,\cot\tfrac{1}{2}\mathrm{V}, \quad \ldots\ldots\ldots\ldots (20)$$

$$\tan\tfrac{1}{2}(c-a) = \tan\tfrac{1}{2}b\,\tan\tfrac{1}{2}\mathrm{V}, \quad \ldots\ldots\ldots\ldots (21)$$

where again the factors on the right are only two in number.

90. We shall now give some examples of the numerical solution of right-angled spherical triangles.

Though in this and in the following chapter seven-figure logarithms are used, it should be borne in mind that for many practical purposes logarithms to four or five figures are quite sufficient. It is useless to introduce into a calculation a degree of accuracy greater than that attained in the measurement of the elements which constitute the data. In navigation, for instance, where angles cannot be measured with anything like the precision of astronomical or geodetical observations, it would be waste of time to employ many-figure logarithms in the calcula-

tions. Of course, with a small number of decimal places, the difficulty of finding a small angle from its cosine, or an angle near a right angle from its sine, will arise more frequently. But the supplemental formulae designed to meet such cases may then be employed, and will give as close an approximation as is necessary.

91. Example 1. *Given*

$$a = 37^\circ\ 48'\ 12'',\ b = 59^\circ\ 44'\ 16'',\ \mathsf{C} = 90^\circ.$$

To find c we have

$$\cos c = \cos a \cos b,$$

$$\mathsf{L}\cos 37^\circ\ 48'\ 12'' = 9{\cdot}8976927$$
$$\mathsf{L}\cos 59^\circ\ 44'\ 16'' = 9{\cdot}7023945$$

$$\mathsf{L}\cos c + 10 = 19{\cdot}6000872$$
$$c = 66^\circ\ 32'\ 6''.$$

To find A and B we have

$$\tan \tfrac{1}{2}(\mathsf{B} - \mathsf{V}) = \tan \tfrac{1}{2}a \tan \tfrac{1}{2}b$$
$$\tan \tfrac{1}{2}(\mathsf{B} + \mathsf{V}) = \cot \tfrac{1}{2}a \tan \tfrac{1}{2}b,$$

where V stands for $90^\circ - \mathsf{A}$.

$$\mathsf{L}\tan \tfrac{1}{2}b = \mathsf{L}\tan 29^\circ\ 52'\ 8'' = 9{\cdot}7591412$$
$$\mathsf{L}\tan \tfrac{1}{2}a = \mathsf{L}\tan 18^\circ\ 54'\ 6'' = 9{\cdot}5345452$$

$$\mathsf{L}\tan \tfrac{1}{2}(\mathsf{B} + \mathsf{V}) = 10{\cdot}2245960$$
$$\mathsf{L}\tan \tfrac{1}{2}(\mathsf{B} - \mathsf{V}) = 9{\cdot}2936864$$

$$\tfrac{1}{2}(\mathsf{B} + \mathsf{V}) = 59^\circ\ 11'\ 45'', \qquad \tfrac{1}{2}(\mathsf{B} - \mathsf{V}) = 11^\circ\ 7'\ 30''.$$
$$\mathsf{B} = 70^\circ\ 19'\ 15'', \qquad \mathsf{V} = 48^\circ\ 4'\ 15'', \qquad \mathsf{A} = 41^\circ\ 55'\ 45''.$$

92. Example 2. *Given*

$$\mathsf{A} = 55^\circ\ 32'\ 45'',\ \mathsf{C} = 90^\circ,\ c = 98^\circ\ 14'\ 24''.$$

To find a we have

$$\sin a = \sin c \sin \mathsf{A},$$

$$\mathsf{L}\sin 98^\circ\ 14'\ 24'' = 9{\cdot}9954932$$
$$\mathsf{L}\sin 55^\circ\ 32'\ 45'' = 9{\cdot}9162323$$

$$\mathsf{L}\sin a + 10 = 19{\cdot}9117255$$
$$a = 54^\circ\ 41'\ 35''.$$

To find B we have

$$\cot \mathsf{B} = \cos c \tan \mathsf{A}.$$

Here $\cos c$ *is negative*; and therefore $\cot \mathsf{B}$ will be negative, and B greater than a right angle. The numerical value of $\cos c$ is the same as that of $\cos 81^\circ\ 45'\ 36''$.

$$\text{L}\cos 81^\circ\ 45'\ 36'' = 9{\cdot}1563065$$
$$\text{L}\tan 55^\circ\ 32'\ 45'' = 10{\cdot}1636102$$

$$\text{L}\cot(180^\circ - \text{B}) + 10 = 19{\cdot}3199167$$
$$180^\circ - \text{B} = 78^\circ\ 12'\ \ 4''$$
$$\text{B} = 101^\circ\ 47'\ 56''.$$

To find b we have $\tan b = \tan c \cos \text{A}.$

Here tan c *is negative*; and therefore $\tan b$ will be negative and b greater than a quadrant.

$$\text{L}\tan 81^\circ\ 45'\ 36'' = 10{\cdot}8391867$$
$$\text{L}\cos 55^\circ\ 32'\ 45'' = 9{\cdot}7526221$$

$$\text{L}\tan(180^\circ - b) + 10 = 20{\cdot}5918088$$
$$180^\circ - b = 75^\circ\ 38'\ 32''$$
$$b = 104^\circ\ 21'\ 28''$$

93. Example 3. *Given*

$$\text{A} = 46^\circ\ 15'\ 25'',\ \ \text{C} = 90^\circ,\ \ a = 42^\circ\ 18'\ 45''.$$

To find c we have

$$\sin c = \frac{\sin a}{\sin \text{A}},$$

$$\text{L}\sin c = 10 + \text{L}\sin a - \text{L}\sin \text{A},$$
$$10 + \text{L}\sin 42^\circ\ 18'\ 45'' = 19{\cdot}8281272$$
$$\text{L}\sin 46^\circ\ 15'\ 25'' = 9{\cdot}8588065$$

$$\text{L}\sin c = 9{\cdot}9693207$$

$$c = 68^\circ\ 42'\ 59''\ \text{ or }\ 111^\circ\ 17'\ 1''.$$

To find b we have $\sin b = \tan a \cot \text{A},$

$$\text{L}\tan 42^\circ\ 18'\ 45'' = 9{\cdot}9591983$$
$$\text{L}\cot 46^\circ\ 15'\ 25'' = 9{\cdot}9809389$$

$$\text{L}\sin b + 10 = 19{\cdot}9401372$$

$$b = 60^\circ\ 36'\ 10''\ \text{ or }\ 119^\circ\ 23'\ 50''.$$

To find B we have $\sin \text{B} = \dfrac{\cos \text{A}}{\cos a},$

$$\text{L}\sin \text{B} = 10 + \text{L}\cos \text{A} - \text{L}\cos a,$$

$$10 + \text{L}\cos 46^\circ\ 15'\ 25'' = 19{\cdot}8397454$$
$$\text{L}\cos 42^\circ\ 18'\ 45'' = 9{\cdot}8689289$$

$$\text{L}\sin \text{B} = 9{\cdot}9708165$$

$$\text{B} = 69^\circ\ 13'\ 47''\ \text{ or }\ 110^\circ\ 46'\ 13''.$$

94. Example 4. (REIDT, § 32, *No.* 12.)

Given $c = 37° \ 40' \ 20''$, $a = 37° \ 40' \ 12''$, $C = 90°$.

$\frac{1}{2}(c+a) = 37° \ 40' \ 16''$,

$L \tan \frac{1}{2}(c+a) = 9{\cdot}88766$

$L \tan \frac{1}{2}(c-a) = 5{\cdot}28763$

$L \tan^2 \frac{1}{2}b = 5{\cdot}17529$

$L \tan^2 (45° - \frac{1}{2}A) = 5{\cdot}39997$

$L \tan \frac{1}{2}b = 7{\cdot}58765$

$L \tan (45° - \frac{1}{2}A) = 7{\cdot}69999$

$\frac{1}{2}b = 0° \ 13' \ 18''$

$45° - \frac{1}{2}A = 0° \ 17' \ 14''$

$\frac{1}{2}(c-a) = 0° \ 0' \ 4''$.

$L \sin (c-a) = 5{\cdot}58866$

$L \sin (c+a) = 9{\cdot}98563$

$L \tan^2 \frac{1}{2}B = 5{\cdot}60303$

$L \tan \frac{1}{2}B = 7{\cdot}80152$

$\frac{1}{2}B = 0° \ 21' \ 46''$

$B = 0° \ 43' \ 32''$

$b = 0° \ 26' \ 36''$

$A = 89° \ 25' \ 32''$

95. Example 5. (REIDT, § 32, *No.* 6.)

Given $a = 34° \ 6' \ 13''$, $A = 34° \ 7' \ 41''$, $C = 90°$.

$A - a = 0° \ 1' \ 28''$

$\frac{1}{2}(A-a) = 0° \ 0' \ 44''$

$L \tan \frac{1}{2}(A-a) = 6{\cdot}32903$

$L \tan \frac{1}{2}(A+a) = 9{\cdot}83088$

$L \tan^2 (45° - \frac{1}{2}c) = 6{\cdot}49815$

$L \tan^2 (45° - \frac{1}{2}B) = 6{\cdot}15991$

$L \tan (45° - \frac{1}{2}c) = 8{\cdot}24908$

$L \tan (45° - \frac{1}{2}B) = 8{\cdot}07996$

$45° - \frac{1}{2}c = 1° \ 1' \ 0''$

$45° - \frac{1}{2}B = 0° \ 41' \ 19''$

$A + a = 68° \ 13' \ 54''$

$\frac{1}{2}(A+a) = 34° \ 6' \ 57''$

$L \sin (A-a) = 6{\cdot}63006$

$L \sin (A+a) = 9{\cdot}96788$

$L \tan^2 (45° - \frac{1}{2}b) = 6{\cdot}66218$

$L \tan (45° - \frac{1}{2}b) = 8{\cdot}33109$

$45° - \frac{1}{2}b = 1° \ 13' \ 40''$

$b = 87° \ 32' \ 40''$

$c = 87° \ 58' \ 0''$

$B = 88° \ 37' \ 22''$

EXAMPLES III.

Prove the relations contained in Examples 1 to 5 for a triangle ABC in which the angle C is a right angle.

1. $\sin^2 \frac{1}{2}c = \sin^2 \frac{1}{2}a \cos^2 \frac{1}{2}b + \cos^2 \frac{1}{2}a \sin^2 \frac{1}{2}b$.

2. $\tan \frac{1}{2}(c+a) \tan \frac{1}{2}(c-a) = \tan^2 \frac{1}{2}b$.

3. $\sin (c-b) = \tan^2 \frac{1}{2}A \sin (c+b)$.

4. $\sin a \tan \frac{1}{2}A - \sin b \tan \frac{1}{2}B = \sin (a-b)$.

5. $$\sin (c-a) = \sin b \cos a \tan \tfrac{1}{2}B,$$
$$\sin (c-a) = \tan b \cos c \tan \tfrac{1}{2}B.$$

6. If ABC be a spherical triangle, right-angled at C, and $\cos A = \cos^2 a$, shew that if A be not a right angle, $b+c=\frac{1}{2}\pi$ or $\frac{3}{2}\pi$, according as b and c are both less or both greater than $\frac{1}{2}\pi$.

7. If α, β be the arcs drawn from the right angle respectively perpendicular to and bisecting the hypotenuse c, shew that

$$\sin^2 \tfrac{1}{2}c(1+\sin^2\alpha)=\sin^2\beta.$$

8. In a triangle, if C be a right angle and D the middle point of AB, shew that

$$4\cos^2\tfrac{1}{2}c\sin^2 CD=\sin^2 a+\sin^2 b.$$

9. In a right-angled triangle, if δ be the length of the arc drawn from C perpendicular to the hypotenuse AB, shew that

$$\cot\delta=\sqrt{(\cot^2 a+\cot^2 b)}.$$

10. OAA_1 is a spherical triangle right-angled at A_1 and acute-angled at A; the arc A_1A_2 of a great circle is drawn perpendicular to OA, then A_2A_3 is drawn perpendicular to OA_1, and so on: shew that A_nA_{n+1} vanishes when n becomes infinite; and find the value of $\cos AA_1 \cos A_1A_2 \cos A_2A_3 \ldots\ldots$ to infinity.

11. ABC is a right-angled spherical triangle, A not being the right angle: shew that, if $A=a$, then c and b are quadrants.

12. If δ be the length of the arc drawn from C perpendicular to AB in *any* triangle, shew that

$$\cos\delta=\operatorname{cosec} c(\cos^2 a+\cos^2 b-2\cos a\cos b\cos c)^{\frac{1}{2}}.$$

13. ABC is a great circle of a sphere; AA′, BB′, CC′ are arcs of great circles drawn at right angles to ABC and reckoned positive when they lie on the same side of it: shew that the condition that A′, B′, C′ should lie in a great circle is

$$\tan AA'\sin BC+\tan BB'\sin CA+\tan CC'\sin AB=0.$$

14. Perpendiculars are drawn from the angles A, B, C of any triangle, meeting the opposite sides at D, E, F respectively: shew that

$$\tan BD\tan CE\tan AF=\tan DC\tan EA\tan FB.$$

15. Ox, Oy are two great circles of a sphere at right angles to each other, P is any point in AB another great circle. $OC(=p)$ is the arc perpendicular to AB from O, making the angle $COx(=\alpha)$ with Ox. PM, PN are arcs perpendicular to Ox, Oy respectively: shew that if $OM=x$ and $ON=y$,

$$\cos\alpha\tan x+\sin\alpha\tan y=\tan p.$$

16. The position of a point on a sphere, with reference to two great circles at right angles to each other as axes, is determined by the portions θ, ϕ of these circles cut off by great circles through the point, and through two points on the axes, each $\frac{1}{2}\pi$ from their point of intersection: shew that if the three points (θ, ϕ), (θ', ϕ'), (θ'', ϕ'') lie on the same great circle

$$\tan\phi(\tan\theta' - \tan\theta'') + \tan\phi'(\tan\theta'' - \tan\theta) + \tan\phi''(\tan\theta - \tan\theta') = 0.$$

17. If a point on a sphere be referred to two great circles at right angles to each other as axes, by means of the portions of these axes cut off by great circles drawn through the point and two points on the axes each 90° from their intersection, shew that the equation to a great circle is

$$\tan\theta\cot\alpha + \tan\phi\cot\beta = 1.$$

18. In a spherical triangle, if $A = \frac{\pi}{5}$, $B = \frac{\pi}{3}$, and $C = \frac{\pi}{2}$, shew that $a + b + c = \frac{\pi}{2}$.

EXAMPLES IV.

Solve the triangle in the following cases:

1.	Given	$b = 137° \; 3' \; 48''$,	$A = 147° \; 2' \; 54''$,	$C = 90°$.
	Results:	$c = 47° \; 57' \; 15''$,	$a = 156° \; 10' \; 34''$,	$B = 113° \; 28'$.
2.	Given	$c = 61° \; 4' \; 56''$,	$a = 40° \; 31' \; 20''$,	$C = 90°$.
	Results:	$b = 50° \; 30' \; 29''$,	$B = 61° \; 50' \; 28''$,	$A = 47° \; 54' \; 21''$.
3.	Given	$A = 36°$,	$B = 60°$,	$C = 90°$.
	Results:	$a = 20° \; 54' \; 18''{\cdot}5$,	$b = 31° \; 43' \; 3''$,	$c = 37° \; 21' \; 38''{\cdot}5$.
4.	Given	$a = 59° \; 28' \; 27''$,	$A = 66° \; 7' \; 20''$,	$C = 90°$.
	Results:	$c = 70° \; 23' \; 42''$,	$b = 48° \; 39' \; 16''$,	$B = 52° \; 50' \; 20''$.
	or,	$c = 109° \; 36' \; 18''$,	$b = 131° \; 20' \; 44''$,	$B = 127° \; 9' \; 40''$.

EXAMPLES V.

1. If ABC be a triangle in which the angle C is a right angle, prove the following relations:

(1) $\sin^2 a + \sin^2 b - \sin^2 c = \sin^2 a \sin^2 b$.

(2) $\cos^2 A \sin^2 c = \sin(c - a)\sin(c + a)$.

(3) $\sin^2 A \cos^2 c = \sin(A - a)\sin(A + a)$.

(4) $\cos^2 A + \cos^2 c - \cos^2 a = \cos^2 A \cos^2 c$.

(5) $\sin(a + b)\tan\frac{1}{2}(A + B) = \sin(a - b)\cot\frac{1}{2}(A - B)$.

(6) $$\sin(\mathsf{A}+\mathsf{B})=\frac{\cos b+\cos a}{1+\cos b\cos a},$$

$$\sin(\mathsf{A}-\mathsf{B})=\frac{\cos b-\cos a}{1-\cos b\cos a}.$$

2. If CD be the great circle drawn through C perpendicular to the hypotenuse AB,

$$\sin^2 \mathsf{CD}=\tan \mathsf{AD}\tan \mathsf{DB}.$$

3. A ship starts from a point on the equator and sails in a great circle, cutting the equator at an angle of 45°: find how much she has changed her longitude when she has reached a latitude $\tan^{-1}(\frac{1}{2})$.

(R. U. I., 1898.)

4. A and B are two places in the northern hemisphere, whose latitudes are λ and λ', and the difference of their longitudes l (where l is supposed less than 90°); shew that, if a ship sailing by the shortest course from A to B increases her latitude the whole way, $\tan\lambda\cot\lambda'$ must not be greater than $\cos l$. (R. U. I., 1898.)

CHAPTER V.

SOLUTION OF OBLIQUE-ANGLED TRIANGLES.

96. The solution of oblique-angled triangles may be made in some cases to depend immediately on the solution of right-angled triangles; we shall indicate these cases before considering the subject generally.

(1) Suppose a triangle to have one of its given sides equal to a *quadrant.* In this case the polar triangle has its corresponding angle a right angle; the polar triangle can therefore be solved by the rules of the preceding Chapter, and thus the elements of the primitive triangle become known.*

(2) Suppose among the given elements of a triangle there are two *equal sides* or two *equal angles.* By drawing an arc from the vertex to the middle point of the base, the triangle is divided into two equal *right-angled* triangles; by the solution of one of these right-angled triangles the required elements can be found.

(3) Suppose among the given elements of a triangle there are two sides, one of which is the supplement of the other, or two angles, one of which is the supplement of the other.

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* **Quadrantal triangle.** The following are the formulae for the quadrantal triangle in which the side c is a quadrant:

$$\cos C + \cos A \cos B = 0 \quad \text{......... (1).}$$

$$\left.\begin{aligned} \sin B &= \sin b \sin C \\ \sin A &= \sin a \sin C \end{aligned}\right\} \quad \text{......... (2).}$$

$$\left.\begin{aligned} \tan A + \cos b \tan C &= 0 \\ \tan B + \cos a \tan C &= 0 \end{aligned}\right\} \quad \text{...... (3).}$$

$$\left.\begin{aligned} \tan B &= \tan b \sin A \\ \tan A &= \tan a \sin B \end{aligned}\right\} \quad \text{......... (4).}$$

$$\cos C + \cot a \cot b = 0 \quad \text{......... (5).}$$

$$\left.\begin{aligned} \cos b &= \sin a \cos B \\ \cos a &= \sin b \cos A \end{aligned}\right\} \quad \text{......... (6).}$$

Suppose, for example, that $b+c=\pi$, or else that $B+C=\pi$; produce BA and BC to meet at B′ (see the first figure to Art. 43); then the triangle B′AC has two equal sides given, or else two equal angles given; and by the preceding case its solution can be made to depend on the solution of a right-angled triangle.

We now proceed to the solution of oblique-angled triangles in general. There will be six cases to consider.

97. Case I.—*Having given the three sides.*

Here we have $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$, and similar formulae for cos B and cos C. These formulae, however, are not in a form adapted to logarithms, and would require to be modified by the introduction of an auxiliary angle. Thus we may define ϕ by the relation $\tan\phi = \cos c \sin b \sec a$, and the formula for cos A then reduces to

$$\cos A = \frac{\cot c \sin (b-\phi)}{\sin b \sin \phi}. \qquad (1)$$

But this is an unnecessarily lengthy way of arriving at the desired results, and it is much better to use some one of the formulae already adapted to logarithms which we have in the expressions for the sine, cosine, and tangent of half an angle given in Art. 50. In selecting a formula, attention should be paid to the remarks in *Plane Trigonometry*, Chap. XII, towards the end.

In navigation, the formula

$$\sin^2 \tfrac{1}{2}A = \frac{\sin (s-b) \sin (s-c)}{\sin b \sin c} \qquad (2)$$

is always used; and there are specially prepared tables* giving the logarithm of $\sin^2\frac{1}{2}A$, that is, the halved versine (or *haversine*, as it is called) of A for values of A corresponding to every fifteen seconds of arc. An approximate calculation can thus be effected with great rapidity.

* See INMAN'S *Nautical Tables*, or RAPER'S *Practice of Navigation.*

When, however, haversine tables are not available, it is just as convenient to use the formula

$$\tan \tfrac{1}{2}A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}; \quad \ldots\ldots\ldots\ldots\ldots(3)$$

and when it is required to find not *one* only, but *all* the angles of the triangle, the tangent expression should always be used, as it involves looking up the smallest number of logarithms. The method of procedure is as follows:

Put
$$\tan r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}. \quad \ldots\ldots\ldots\ldots\ldots(4)$$

It will be seen afterwards that r, as here defined, is the angular radius of the inscribed circle; but its interpretation does not concern us at present. Determine the logarithm of $\tan r$, and then use the formulae

$$\tan \tfrac{1}{2}A = \frac{\tan r}{\sin(s-a)}, \quad \tan \tfrac{1}{2}B = \frac{\tan r}{\sin(s-b)}, \quad \tan \tfrac{1}{2}C = \frac{\tan r}{\sin(s-c)}. \quad \ldots(5)$$

98. Numerical Example. Take, for example, the triangle in which we have given

$$a = 70^\circ\ 14'\ 20''$$
$$b = 49^\circ\ 24'\ 10''$$
$$c = 38^\circ\ 46'\ 10''.$$

The calculation is as follows:

$$2s = 158^\circ\ 24'\ 40''$$
$$s = 79^\circ\ 12'\ 20''$$
$$s-a = 8^\circ\ 58'\ 0''$$
$$s-b = 29^\circ\ 48'\ 10''$$
$$s-c = 40^\circ\ 26'\ 10''$$

$L\sin(s-a) = 9{\cdot}1927342$	$L\tan\tfrac{1}{2}A = 10{\cdot}1616832$
$L\sin(s-b) = 9{\cdot}6963704$	$L\tan\tfrac{1}{2}B = 9{\cdot}6580470$
$L\sin(s-c) = 9{\cdot}8119768$	$L\tan\tfrac{1}{2}C = 9{\cdot}5424406$
$8{\cdot}7010814$	$\tfrac{1}{2}A = 55^\circ\ 25'\ 38''$
$L\sin s = 9{\cdot}9922465$	$\tfrac{1}{2}B = 24^\circ\ 28'\ 2''$
	$\tfrac{1}{2}C = 19^\circ\ 13'\ 24''$
$L\tan^2 r = 8{\cdot}7088349$	
$L\tan r = 9{\cdot}3544174$	$A = 110^\circ\ 51'\ 16''$
	$B = 48^\circ\ 56'\ 4''$
	$C = 38^\circ\ 26'\ 48''$

99. Case II.—*Having given the three angles.*

Here we have $\cos a = \dfrac{\cos \mathsf{A} + \cos \mathsf{B} \cos \mathsf{C}}{\sin \mathsf{B} \sin \mathsf{C}}$, with similar formulae for $\cos b$ and $\cos c$, and this may be adapted to logarithms by introducing an angle ϕ such that $\tan \phi = \cos \mathsf{C} \sin \mathsf{B} \sec \mathsf{A}$;

then $$\cos a = \frac{\cot \mathsf{C} \sin(\mathsf{B} + \phi)}{\sin \mathsf{B} \sin \phi}. \qquad (6)$$

It is better, however, to use some one of the expressions for the sine, cosine, and tangent of half a side, given in Art. 56. Thus a single side may be conveniently calculated from the formula

$$\tan \tfrac{1}{2} a = \sqrt{-\frac{\cos \mathsf{S} \cos(\mathsf{S} - \mathsf{A})}{\cos(\mathsf{S} - \mathsf{B}) \cos(\mathsf{S} - \mathsf{C})}}. \qquad (7)$$

When all three sides are required, the shortest method is as follows. First define $\cot \mathsf{R}$ and calculate its logarithm by the relation

$$\cot^2 \mathsf{R} = -\frac{\cos(\mathsf{S} - \mathsf{A}) \cos(\mathsf{S} - \mathsf{B}) \cos(\mathsf{S} - \mathsf{C})}{\cos \mathsf{S}}, \qquad (8)$$

and then find a, b, c from the formulae

$$\cot \tfrac{1}{2} a = \frac{\cot \mathsf{R}}{\cos(\mathsf{S} - \mathsf{A})}, \quad \cot \tfrac{1}{2} b = \frac{\cot \mathsf{R}}{\cos(\mathsf{S} - \mathsf{B})}, \quad \cot \tfrac{1}{2} c = \frac{\cot \mathsf{R}}{\cos(\mathsf{S} - \mathsf{C})}. \qquad (9)$$

The work should be arranged as in the numerical example of the previous Article. The similarity of these two cases to one another is obvious. It will be shewn later (Art. 122) that R is the angular radius of the circumscribed circle of the triangle.

100. There is no ambiguity in the two preceding cases; the triangles, however, may be impossible with the given elements.

101. Case III.—*Having given two sides and the included angle* (a, C, b).

By NAPIER'S analogies

$$\tan \tfrac{1}{2}(\mathsf{A} + \mathsf{B}) = \frac{\cos \tfrac{1}{2}(a - b)}{\cos \tfrac{1}{2}(a + b)} \cot \tfrac{1}{2} \mathsf{C}, \qquad (10)$$

$$\tan \tfrac{1}{2}(\mathrm{A} - \mathrm{B}) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \tfrac{1}{2}\mathrm{C} \, ; \quad \ldots\ldots\ldots\ldots(11)$$

these determine $\frac{1}{2}(\mathrm{A}+\mathrm{B})$ and $\frac{1}{2}(\mathrm{A}-\mathrm{B})$, and thence A and B.

Then c may be found from the formula $\sin c = \dfrac{\sin a \sin \mathrm{C}}{\sin \mathrm{A}}$; in this case, since c is found from its sine, it may be uncertain which of two values is to be given to it; the point may sometimes be settled by observing that the greater side of a triangle is opposite to the greater angle. Or we may determine c from any one of DELAMBRE'S analogies, for instance equation (45) of Art. 63,

$$\cos \tfrac{1}{2}c = \frac{\cos \frac{1}{2}(a+b) \sin \frac{1}{2}\mathrm{C}}{\cos \frac{1}{2}(\mathrm{A}+\mathrm{B})}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

which gives c without ambiguity.

102. Numerical Example. Take, for example, the triangle in which we have given

$$a = 68° \, 20' \, 25'', \quad b = 52° \, 18' \, 15'', \quad \mathrm{C} = 117° \, 12' \, 20''.$$

The calculation is as follows:

$$\tfrac{1}{2}(a-b) = 8° \, 1' \, 5'', \quad \tfrac{1}{2}(a+b) = 60° \, 19' \, 20'', \quad \tfrac{1}{2}\mathrm{C} = 58° \, 36' \, 10''.$$

$\mathrm{L}\sin \frac{1}{2}(a-b) = 9 \cdot 1445280$	$\mathrm{L}\cos \frac{1}{2}(a-b) = 9 \cdot 9957335$
$\mathrm{L}\sin \frac{1}{2}(a+b) = 9 \cdot 9389316$	$\mathrm{L}\cos \frac{1}{2}(a+b) = 9 \cdot 6947120$
$9 \cdot 2055964$	$10 \cdot 3010215$
$\mathrm{L}\cot \frac{1}{2}\mathrm{C} = 9 \cdot 7855690$	$\mathrm{L}\sin \frac{1}{2}\mathrm{C} = 9 \cdot 9312422$
$\mathrm{L}\tan \frac{1}{2}(\mathrm{A}-\mathrm{B}) = 8 \cdot 9911654$	$9 \cdot 6259542$
$\mathrm{L}\tan \frac{1}{2}(\mathrm{A}+\mathrm{B}) = 10 \cdot 0865905$	$\mathrm{L}\cos \frac{1}{2}(\mathrm{A}+\mathrm{B}) = 9 \cdot 8019015$
$\frac{1}{2}(\mathrm{A}-\mathrm{B}) = 5° \, 35' \, 47''$	$\mathrm{L}\cos \frac{1}{2}c = 9 \cdot 8240527$
$\frac{1}{2}(\mathrm{A}+\mathrm{B}) = 50° \, 40' \, 28''$	$\frac{1}{2}c = 48° \, 10' \, 22''$
$\mathrm{A} = 56° \, 16' \, 15''$	$c = 96° \, 20' \, 44''$
$\mathrm{B} = 45° \, 4' \, 41''$	

103. We can also find c, without previously determining A and B, from the formula

$$\cos c = \cos a \cos b + \sin a \sin b \cos \mathrm{C}, \quad \ldots\ldots\ldots\ldots (13)$$

which is free from ambiguity. This formula may be adapted to logarithms * by introducing an auxiliary angle θ defined by the relation

$$\tan\theta = \tan b \cos C; \quad \ldots\ldots\ldots\ldots(14)$$

the expression for c then becomes

$$\cos c = \cos b(\cos a + \sin a \tan\theta) = \frac{\cos b \cos(a-\theta)}{\cos\theta}, \quad \ldots(15)$$

One of the angles may at the same time be determined without ambiguity; for if we use the second of the formulae of Art. 49 we get

$$\cot B = \frac{1}{\sin C}\{\cot b \sin a - \cos a \cos C\} \ldots\ldots\ldots(16)$$

$$= \frac{\cot C}{\sin\theta}\sin(a-\theta). \quad \ldots\ldots\ldots\ldots(17)$$

104. Numerical Example. If, for example, we consider the triangle which we have just solved otherwise, our first step is to determine θ from the formula $\tan\theta = \tan b \cos C$.

Here $\cos C$ *is negative*, and therefore $\tan\theta$ will be negative, and θ greater than a right angle. The numerical value of $\cos C$ is the same as that of $\cos 62° \, 47' \, 40''$.

$$\begin{aligned} L\tan b &= 10{\cdot}1119488 \\ L\cos 62° \, 47' \, 40'' &= 9{\cdot}6600912 \\ \hline L\tan(180° - \theta) + 10 &= 19{\cdot}7720400 \\ 180° - \theta &= 30° \, 36' \, 33'', \\ \text{therefore } \theta &= 149° \, 23' \, 27''. \end{aligned}$$

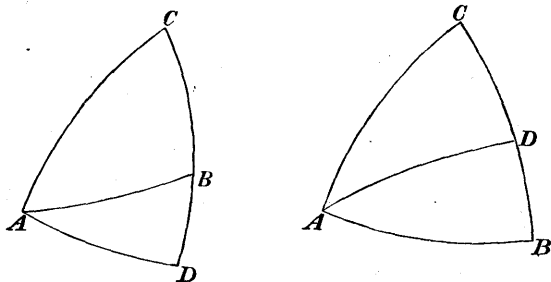
Next we determine c from the formula $\cos c = \dfrac{\cos b \cos(a-\theta)}{\cos\theta}$.

Here $\cos\theta$ *is negative*, and therefore $\cos c$ will be negative, and c will be greater than a right angle. The numerical value of $\cos\theta$ is the same as that of $\cos(180° - \theta)$, that is, of $\cos 30° \, 36' \, 33''$; and the value of $\cos(a-\theta)$ is the same as that of $\cos(\theta - a)$, that is, of $\cos 81° \, 3' \, 2''$.

* Compare formulae (22) and (23) of Art. 53.

$$
\begin{aligned}
\mathsf{L}\cos b &= 9{\cdot}7863748 \\
\mathsf{L}\cos 81^\circ\, 3'\, 2'' &= \underline{9{\cdot}1919060} \\
& \quad 18{\cdot}9782808 \\
\mathsf{L}\cos 30^\circ\, 36'\, 33'' &= \underline{9{\cdot}9348319} \\
\mathsf{L}\cos(180^\circ - c) &= 9{\cdot}0434489 \\
180^\circ - c &= 83^\circ\, 39'\, 17'' \\
c &= 96^\circ\, 20'\, 43''
\end{aligned}
$$

Thus by taking only the nearest number of seconds in the tables the two methods give values of c which differ by $1''$; if, however, we estimate fractions of a second, the methods will agree in giving about $43\frac{1}{2}$ as the number of seconds.

105. The method of Art. 103 is really equivalent to resolving the triangle into the sum or difference of two right-angled triangles.

From A draw the arc AD perpendicular to CB or CB produced; then, by Art. 73, $\tan \text{CD} = \tan b \cos \text{C}$, and this determines CD, and then DB is known. Again, by Art. 73,

$$\cos c = \cos \text{AD} \cos \text{DB} = \cos \text{DB} \frac{\cos b}{\cos \text{CD}};$$

this finds c. It is obvious that CD is what was denoted by θ in Art. 103.

By Art. 73,

$$\tan \text{AD} = \tan \text{C} \sin \text{CD}, \text{ and } \tan \text{AD} = \tan \text{A}\hat{\text{B}}\text{D} \sin \text{DB};$$

thus
$$\tan \text{A}\hat{\text{B}}\text{D} \sin \text{DB} = \tan \text{C} \sin \theta,$$

where $\text{DB} = a - \theta$ or $\theta - a$, according as D is on CB or CB produced,

and $\mathrm{A\hat{B}D}$ is either B or the supplement of B; this formula enables us to find B independently of A.

Thus, in the present case, there is no real ambiguity, and the triangle is always possible.

106. Case IV. *Having given two angles and the included side* (A, c, B).

By NAPIER'S analogies,

$$\tan \tfrac{1}{2}(a+b)=\frac{\cos \frac{1}{2}(\mathrm{A}-\mathrm{B})}{\cos \frac{1}{2}(\mathrm{A}+\mathrm{B})} \tan \tfrac{1}{2}c, \qquad (18)$$

$$\tan \tfrac{1}{2}(a-b)=\frac{\sin \frac{1}{2}(\mathrm{A}-\mathrm{B})}{\sin \frac{1}{2}(\mathrm{A}+\mathrm{B})} \tan \tfrac{1}{2}c\,; \qquad (19)$$

these determine $\frac{1}{2}(a+b)$ and $\frac{1}{2}(a-b)$, and thence a and b.

Then C may be found from the formula $\sin \mathrm{C}=\dfrac{\sin \mathrm{A} \sin c}{\sin a}$; in this case, since C is found from its sine, it may be uncertain which of two values is to be given to it; the point may sometimes be settled by observing that the greater angle of a triangle is opposite to the greater side. Or we may determine C from any one of DELAMBRE'S analogies, for instance equation (45) of Art. 63,

$$\sin \tfrac{1}{2}\mathrm{C}=\frac{\cos \frac{1}{2}(\mathrm{A}+\mathrm{B}) \cos \frac{1}{2}c}{\cos \frac{1}{2}(a+b)}, \qquad (20)$$

which determines C without ambiguity, since $\frac{1}{2}\mathrm{C}$ cannot exceed 90°.

107. Or we may determine C, without previously determining a and b, from the formula $\cos \mathrm{C}=-\cos \mathrm{A} \cos \mathrm{B}+\sin \mathrm{A} \sin \mathrm{B} \cos c$. This formula may be adapted to logarithms thus:

$$\cos \mathrm{C}=\cos \mathrm{B}(-\cos \mathrm{A}+\sin \mathrm{A} \tan \mathrm{B} \cos c);$$

assume $\cot \phi=\tan \mathrm{B} \cos c$; then

$$\cos \mathrm{C}=\cos \mathrm{B}(-\cos \mathrm{A}+\cot \phi \sin \mathrm{A})=\frac{\cos \mathrm{B} \sin (\mathrm{A}-\phi)}{\sin \phi};$$

this is adapted to logarithms.

108. Or we may treat this case conveniently by resolving the triangle into the sum or difference of two right-angled triangles. From A draw the arc AD perpendicular to CB (see the right-hand figure of Art. 105); then, by Art. 73, $\cos c = \cot B \cot D\hat{A}B$, and this determines $D\hat{A}B$, and then $C\hat{A}D$ is known. Again, by Art. 73,

$$\cos AD \sin C\hat{A}D = \cos C, \quad \text{and} \quad \cos AD \sin B\hat{A}D = \cos B\,;$$

therefore $$\frac{\cos C}{\sin CAD} = \frac{\cos B}{\sin BAD}\,; \text{ this gives C.}$$

It is obvious that $D\hat{A}B$ is what was denoted by ϕ in the previous Article.

By Art. 73,

$$\tan AD = \tan AC \cos C\hat{A}D, \quad \text{and} \quad \tan AD = \tan AB \cos B\hat{A}D\,;$$

thus $$\tan b \cos C\hat{A}D = \tan c \cos \phi,$$

where $C\hat{A}D = A - \phi$; this formula enables us to find b independently of a.

Similarly we may proceed when the perpendicular AD falls on CB *produced* (see the left-hand figure of Art. 105).

Thus, in the present case, there is no real ambiguity; moreover the triangle is always possible.

109. Case V.—*Having given two sides and the angle opposite one of them* (a, b, A).

The angle B may be found from the formula

$$\sin B = \frac{\sin b}{\sin a} \sin A\,; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (21)$$

and then C and c may be found from NAPIER'S analogies,

$$\tan \tfrac{1}{2}C = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \tfrac{1}{2}(A-B), \quad \ldots\ldots\ldots\ldots (22)$$

$$\tan \tfrac{1}{2}c = \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a-b). \quad \ldots\ldots\ldots\ldots (23)$$

In this case, since B is found from its sine, there will sometimes be two solutions; and sometimes there will be no solution at all, namely, when the value found for sin B is greater than unity.

110. When two values of B present themselves, in order that either of them should be admissible it is necessary that it should yield positive values of $\tan\frac{1}{2}C$ and $\tan\frac{1}{2}c$, when substituted in equations (22) and (23). Now $\sin\frac{1}{2}(a+b)$ and $\sin\frac{1}{2}(A+B)$ are always positive, and $\frac{1}{2}(a-b)$ and $\frac{1}{2}(A-B)$ are numerically less than $90°$; accordingly what is required reduces simply to this, that $a-b$ and $A-B$ should be of the same sign.

It is not difficult to shew that this *necessary* condition is also a *sufficient* condition for the existence of a triangle having the elements a, b, A, B. Let C and c be as defined by equations (22) and (23). C being by hypothesis less than 180°, a triangle can be constructed having the elements a, b, C; and if its remaining elements be called A', B', c', these must satisfy equations similar in form to equations (21), (22), and (23). Hence it is easy to see that $A'=A$, $B'=B$, $c'=c$, and that therefore the triangle so constructed satisfies the original data.

Thus the problem of finding a triangle with the given elements has two solutions, one solution, or is impossible, according as both, one, or none of the values of B are such as to make $A-B$ and $a-b$ of the same sign. We shall return to this subject, and examine it in detail, in a subsequent Article.

111. Numerical Example. *Given*

$$a=50°\ 45'\ 20'',\quad b=69°\ 12'\ 40'',\quad A=44°\ 22'\ 10''.$$

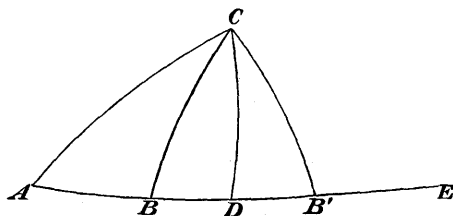
The calculation is as follows:

$$\tfrac{1}{2}(a+b)=59°\ 59'\ 0'',\quad \tfrac{1}{2}(a-b)=-9°\ 13'\ 40''.$$

$$\begin{array}{ll} L\sin b=9{\cdot}9707626 & B_1=\ \ 57°\ 34'\ 51''{\cdot}4 \\ L\sin a=9{\cdot}8889956 & B_2=122°\ 25'\ \ 8''{\cdot}6 \\ \hline \quad 0{\cdot}0817670 & A\ =\ \ 44°\ 22'\ 10'' \\ L\sin A=9{\cdot}8446525 & A-B_1=-13°\ 12'\ 41''{\cdot}4 \\ L\sin B=9{\cdot}9264195 & A-B_2=-78°\ \ 2'\ 58''{\cdot}6 \end{array}$$

There are two solutions, since $A - B_1$, $A - B_2$, and $a - b$ are all negative.

$\frac{1}{2}(B_1 - A) = 6° 36' 20''{\cdot}7$	$\frac{1}{2}(B_2 - A) = 39° 1' 29''{\cdot}3$
$\frac{1}{2}(B_1 + A) = 50° 58' 30''{\cdot}7$	$\frac{1}{2}(B_2 + A) = 83° 23' 39''{\cdot}3$
$L \sin \frac{1}{2}(b - a) = 9{\cdot}2050952$	$L \sin \frac{1}{2}(B_1 + A) = 9{\cdot}8903503$
$L \sin \frac{1}{2}(b + a) = 9{\cdot}9374577$	$L \sin \frac{1}{2}(B_1 - A) = 9{\cdot}0608366$
$9{\cdot}2676375$	$0{\cdot}8295137$
$L \cot \frac{1}{2}(B_1 - A) = 10{\cdot}9362705$	$L \sin \frac{1}{2}(B_2 + A) = 9{\cdot}9971072$
$L \cot \frac{1}{2}(B_2 - A) = 10{\cdot}0912464$	$L \sin \frac{1}{2}(B_2 - A) = 9{\cdot}7991039$
	$0{\cdot}1980033$
	$L \tan \frac{1}{2}(b - a) = 9{\cdot}2107523$
$L \tan \frac{1}{2}C_1 = 10{\cdot}2039080$	$L \tan \frac{1}{2}c_1 = 10{\cdot}0402660$
$L \tan \frac{1}{2}C_2 = 9{\cdot}3588839$	$L \tan \frac{1}{2}c_2 = 9{\cdot}4087556$
$\frac{1}{2}C_1 = 57° 58' 55''{\cdot}3$	$\frac{1}{2}c_1 = 47° 39' 8''{\cdot}3$
$\frac{1}{2}C_2 = 12° 52' 15''{\cdot}8$	$\frac{1}{2}c_2 = 14° 22' 32''{\cdot}4$
$C_1 = 115° 57' 50''{\cdot}6$	$c_1 = 95° 18' 16''{\cdot}6$
$C_2 = 25° 44' 31''{\cdot}6$	$c_2 = 28° 45' 4''{\cdot}8$

112. We might also treat this case conveniently by resolving the triangle into the sum or difference of two right-angled triangles.

Let $CA = b$, and let $C\hat{A}E$ = the given angle A; from C draw CD perpendicular to AE, and let CB and $CB' = a$; thus the figure shews that there may be two triangles which have the given elements. Then, by Art. 73, $\cos b = \cot A \cot A\hat{C}D$; this finds $A\hat{C}D$. Again, by Art. 73,

$$\tan CD = \tan AC \cos A\hat{C}D,$$

and $$\tan CD = \tan CB \cos B\hat{C}D, \text{ or } \tan CB' \cos B'\hat{C}D;$$

therefore $\tan \mathrm{AC} \cos \widehat{\mathrm{ACD}} = \tan \mathrm{CB} \cos \widehat{\mathrm{BCD}}$, or $\tan \mathrm{CB}' \cos \widehat{\mathrm{B'CD}}$; this finds $\widehat{\mathrm{BCD}}$ or $\widehat{\mathrm{B'CD}}$.

Also, by Art. 73, $\tan \mathrm{AD} = \tan \mathrm{AC} \cos \mathrm{A}$; this finds AD. Then

$$\cos \mathrm{AC} = \cos \mathrm{CD} \cos \mathrm{AD},$$

$$\cos \mathrm{CB} = \cos \mathrm{CD} \cos \mathrm{BD},$$

$$\text{or } \cos \mathrm{CB}' = \cos \mathrm{CD} \cos \mathrm{B'D};$$

therefore $$\frac{\cos \mathrm{AC}}{\cos \mathrm{AD}} = \frac{\cos \mathrm{CB}}{\cos \mathrm{BD}} \text{ or } \frac{\cos \mathrm{CB}'}{\cos \mathrm{B'D}};$$

this finds BD or B′D.

113. Reidt's method of solution. When we use the notation

$$\left.\begin{array}{ll} \mathrm{A} + a = 4s, & \mathrm{A} - a = 4d, \\ \mathrm{B} + b = 4s', & \mathrm{B} - b = 4d', \\ \mathrm{C} + c = 4s'', & \mathrm{C} - c = 4d'', \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots (24)$$

we have seen that the first two of REIDT'S analogies (Art. 69, equations (63) and (64)), take the form

$$\tan^2(45^\circ - s'') = \cot(s - s')\tan(s + s')\tan(d - d')\tan(d + d'), \ldots (25)$$

$$\tan^2(45^\circ - d'') = \tan(s - s')\tan(s + s')\cot(d - d')\tan(d + d'). \ldots (26)$$

When, now, two values of B have been determined from the sine formula, we may take one of them, say that which is less than 90°, and substitute its value in the right-hand members of these analogies; from them we can then find the corresponding values of s'' and d'', and thence those of C and c.

The elements C_0 and c_0 of the other triangle, that which has B obtuse, are then derived from equations (67) and (68) of Art. 69, namely,

$$\tan^2 d''_0 = \tan(s - s')\cot(s + s')\tan(d - d')\tan(d + d'), \ldots (27)$$

$$\tan^2 s''_0 = \tan(s - s')\tan(s + s')\tan(d - d')\cot(d + d'). \ldots (28)$$

The great advantage of this method is that, once B has been determined, only four logarithms have to be looked out to complete the solution, instead of six as in the other methods.

114. Numerical example. (REIDT, § 37, No. 3.)

Given $b=70° 40'$, $a=40° 20'$, $A=40°$.

$L \sin b = 9{\cdot}97479$
$L \sin a = 9{\cdot}81106$
$0{\cdot}16373$
$L \sin A = 9{\cdot}80807$
$L \sin B = 9{\cdot}97180$

$B_1 = 69° 34' 30''$
$B_2 = 110° 25' 30''$

$s' = 35° 3' 37''$ | $s - s' = -14° 58' 37''$
$d' = -0° 16' 22''$ | $d - d' = 0° 11' 22''$
$s = 20° 5'$ | $s + s' = 55° 8' 37''$
$d = -0° 5'$ | $d + d' = -0° 21' 22''$

$L \tan(s - s') = 9{\cdot}42736$ (neg.) | $45° - s'' = 0° 36' 7''$
$L \tan(d - d') = 7{\cdot}51968$ | $45° - d'' = 40° 21' 2''$
$L \tan(s + s') = 10{\cdot}15710$ | $90° - 2s'' = 1° 12' 14''$
$L \tan(d + d') = 7{\cdot}79364$ (neg.) | $90° - 2d'' = 80° 42' 4''$

$L \tan^2(45° - s'') = 6{\cdot}04306$ | $2s'' = 88° 47' 46''$
$L \tan^2(45° - d'') = 9{\cdot}85842$ | $2d'' = 9° 17' 56''$

$L \tan^2 d''_0 = 4{\cdot}58358$ | $C = 98° 5' 42''$
$L \tan^2 s''_0 = 9{\cdot}31050$ | $c = 79° 29' 50''$

$L \tan(45° - s'') = 8{\cdot}02153$ | or $d''_0 = 0° 6' 44''$
$L \tan(45° - d'') = 9{\cdot}92921$ | $s''_0 = 24° 19' 42''$
$L \tan d_0'' = 7{\cdot}29179$ | $2d''_0 = 0° 13' 28''$
$L \tan s_0'' = 9{\cdot}65525$ | $2s''_0 = 48° 39' 24''$

$C_0 = 48° 52' 52''$
$c_0 = 48° 25' 56''$

115. If B cannot be determined with sufficient accuracy from its sine, the following formula may be used :

$$\sin a \sin^2(45° - \tfrac{1}{2}B) = \cos \tfrac{1}{2}(a+b) \sin \tfrac{1}{2}(a-b) + \sin b \sin^2(45° - \tfrac{1}{2}A). \quad \ldots (29)$$

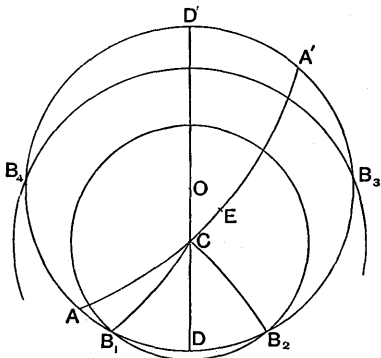
116. Case VI.—*Having given two angles and the side opposite one of them* (A, B, a).

This case is analogous to that immediately preceding, and gives rise to the same ambiguities. The side b may be found from the formula

$$\sin b = \frac{\sin B \sin a}{\sin A}; \quad \ldots\ldots\ldots\ldots (30)$$

and then C and c may be found from NAPIER'S analogies, or preferably from REIDT'S analogies. The formulae are the same as those used in solving *Case* V.

117. The ambiguous case. We now return to the consideration of the ambiguity which may occur in the fifth case of this Chapter, namely, when two sides are given and the angle opposite to one of them. We want to learn how we can infer from inspection of the given elements whether we are to expect two solutions, one solution, or none.

The simplest way of making the investigation is to regard the construction of the triangle as a problem in practical geometry. A straightforward method at once presents itself, which *must* effect the desired construction if the triangle be a possible one; and when this method fails the significance of the failure becomes apparent.

The given elements being A, b, a, we may, since we are concerned only with the shape and not with the position of the triangle, take any great circle ADA′D′ as one of the great circles forming the angle A, and any point A on it as the vertex of that angle. We then draw the great circle AEA′, making the required angle with the first, and measure on it

the arc AC equal to the given value of b. A and C are two corners of the triangle, and we want to find the third corner B, which we know lies somewhere on the first great circle.

Now the angular distance between B and C is a known quantity, namely the given value of a; and therefore B must lie on the small circle whose centre is C and radius a. Accordingly, to construct the triangle, it is only necessary to describe this small circle, for B must be one or other of its points of intersection with the great circle ADA′D′.

The construction altogether fails when the two circles do not meet, and then no triangle exists with the given elements. This happens when the radius a of the small circle is less than the least, or greater than the greatest, angular distance of the point C from the great circle ADA′D′. Now we know from Art. 68 that, if DCOD′ be the great circle joining C to the pole O of the first great circle, and if C lie between O and D, CD is the least and CD′ the greatest arc drawn from C to the circumference of that circle. The angles at D and D′ are right angles, and from the triangle ACD it is seen that $\sin \text{CD} = \sin b \sin \text{A}$, while the sine of the supplement CD′ has the same value. Thus there is no solution when a is greater than the greater, or less than the less, of the two supplementary arcs whose sine equals $\sin b \sin \text{A}$, or in other words, when $\sin a$ is less than $\sin b \sin \text{A}$.

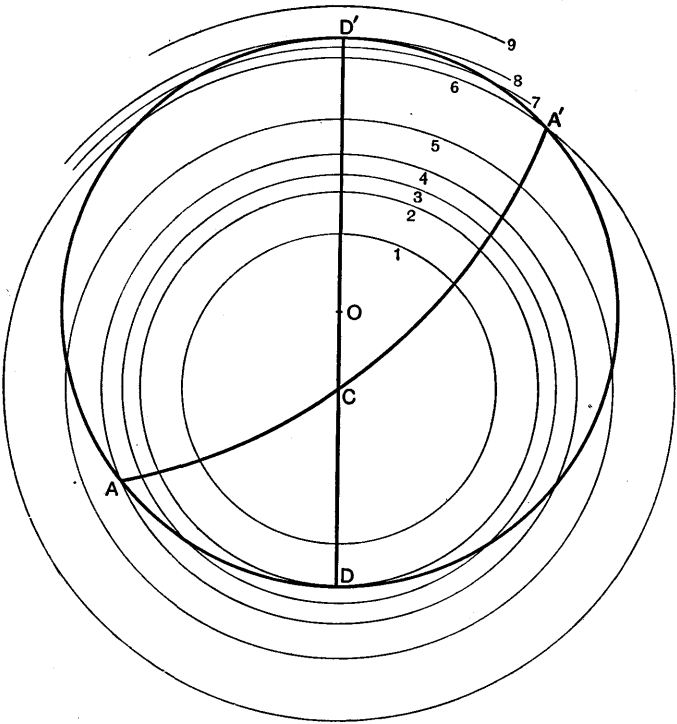
When the circles do intersect, say in B_1 and B_2, the triangles AB_1C, AB_2C may be solutions of the problem, or they may not. This depends on whether they satisfy the restriction contained in the definition of a spherical triangle, that each side shall be less than a semicircle. If the side AB_1, *measured from* A *along that one of the arcs* ADA′, AD′A′ *which is an arm of the given angle* A, be less than two right angles, then AB_1C is a triangle satisfying the data; but if the arc AB_1, so measured, be greater than a semicircle, then the triangle AB_1C is not a solution of the problem. The same test must be applied to B_2.

For example, in the above figure, if A be an acute angle it is the angle between the arcs AEA′ and ADA′, and therefore B_1 or B_2 will yield a solution only if it lie on the arc ADA′. In the figure as drawn both B_1 and B_2 comply with this condition, and so there are two solutions. If the intersections of the great and small circles were situated as B_3 and B_4, then B_3 would give a solution, but B_4 would not. On the other hand if the given value of A be obtuse, it is contained by the arcs AEA′, AD′A′, and so positions of B are only admissible provided they lie on the arc AD′A′. Thus if B_3 and B_4 be the intersections of the great and small circles, there is only one solution, namely that corresponding to B_4; if B_1 and B_2 were the intersections of the circles there would be no solution, since both these points are now inadmissible.

If B_1 coincided with A the triangle would vanish, if B_2 coincided with A′ the triangle would become a lune; neither of these can be classed as a solution. Coincident solutions occur when B_1, B_2, being both admissible, coincide; this would be the case if A were acute and a equal to CD, or if A were obtuse and a equal to CD′. There is an infinite number of solutions when the great and small circles coincide, that is, when C is at O, and a is a right angle.

It will now be easy, by means of the following figure, to examine every case that can arise, and determine the number of solutions which it admits of.

The more lightly drawn circles in the diagram are the small circles having centre C and radius b; there are nine of them, typical of nine values of b and representing nine different cases. Each circle is denoted by a number, and the same number is prefixed to the corresponding case in the catalogue of results. For any particular case we have only to look at the circle bearing the corresponding number, see in how many points it meets the great circle ADA′D′, and whether such points

lie on the particular semicircle on which admissible values of B must be situated.

I. Let $b < 90°$.

(α) If A $< 90°$.

Admissible positions of B are confined to ADA′.

(1) $a <$ CD, - - - - - - no solution.
(2) $a =$ CD, - - - - two coincident solutions.
(3) $a >$ CD, $< b$, - - - - - two solutions.
(4, 5) $a \geq b$, $< 180° - b$, - - - one solution.
(6, 7, 8, 9) $a \geq 180° - b$, - - - no solution.

(β) If $A = 90°$.

In this class of cases A coincides with D, and $CD = b$. Either of the semi-circles AA′ may be taken as the range of admissible positions of B, but not both.

(1, 2, 4) $a \leq b$, - - - - - - no solution.
(5) $a > b$, $< 180 - b$, - - - - one solution.
(6, 8, 9), $a \geq 180° - b$, - - - no solution.

(γ) If $A > 90°$.

Admissible positions of B lie only on AD′A′.

(1, 2, 3, 4) $a \leq b$, - - - - no solution.
(5, 6) $a > b$, $\leq 180° - b$, - - - one solution.
(7) $a > 180° - b$, $< CD'$, - - - two solutions.
(8) $a = CD'$, - - - - two coincident solutions.
(9) $a > CD'$, - - - - - - no solution.

II. Let $b > 90°$.

We now take the vertex of the triangle at A′ instead of A, and b is the arc A′C.

The table of results is derived from that of Class I. by substituting $180° - b$ for b, and b for $180° - b$.

III. Let $b = 90°$.

The arcs CA, CA′ are now equal, and circles 4, 5, and 6 become coincident. $A = CD$ or CD', according as it is acute or obtuse.

(α) If $A < 90°$.

(1) $a < A$, - - - - - - no solution.
(2) $a = A$, - - - - two coincident solutions.
(3) $a > A$, $< 90°$, - - - - - two solutions.
(6, 7, etc.) $a \geq 90°$ - - - - no solution.

(β) If $A = 90°$.

C coincides with O. Circles 2, 3, 4, 5, 6, 7, 8 coincide with the great circle.

(1) $a < 90°$, - - - - - - no solution.
(2, 8) $a = 90°$, - - infinite number of solutions.
(9) $a > 90°$, - - - - - - no solution.

(γ) If $A > 90°$.

(1, 6) $a \leq b$, - - - - - no solution.
(7) $a > b$, $< A$, - - - - - two solutions.
(8) $a = A$, - - - - two coincident solutions.
(9) $a > A$, - - - - - - no solution.

The student will find it advantageous to make a special diagram for Class III.

118. The ambiguities which occur in the last case in the solution of oblique-angled triangles may be deduced from those of the previous case by means of the polar triangle.

EXAMPLES VI.

1. The sides of a triangle are 105°, 90°, and 75° respectively: find the sines of all the angles.

2. Shew that $\tan \frac{1}{2}A \tan \frac{1}{2}B = \frac{\sin(s-c)}{\sin s}$. Solve a triangle when a side, an adjacent angle, and the sum of the other two sides are given.

3. Solve a triangle having given a side, an adjacent angle, and the sum of the other two angles.

4. A triangle has the sum of two sides equal to a semi-circumference: find the arc joining the vertex with the middle of the base.

5. If a, b, c are known, c being a *quadrant*, determine the angles: shew also that if δ be the perpendicular on c from the opposite angle, $\cos^2\delta = \cos^2 a + \cos^2 b$.

6. If one side of a spherical triangle be divided into four equal parts, and θ_1, θ_2, θ_3, θ_4 be the angles subtended at the opposite corner by the parts taken in order, shew that

$$\sin(\theta_1 + \theta_2)\sin\theta_2 \sin\theta_4 = \sin(\theta_3 + \theta_4)\sin\theta_1 \sin\theta_3.$$

7. In a spherical triangle if $A = B = 2C$, shew that

$$8\sin(a + \tfrac{1}{2}c)\sin^2\tfrac{1}{2}c\cos\tfrac{1}{2}c = \sin^3 a.$$

8. In a spherical triangle if $A = B = 2C$, shew that

$$8\sin^2\tfrac{1}{2}C(\cos s + \sin\tfrac{1}{2}C)\cos\tfrac{1}{2}c = \cos a.$$

9. If the equal sides of an isosceles triangle ABC be bisected by an arc DE, and BC be the base, shew that

$$\sin \tfrac{1}{2}\text{DE} = \tfrac{1}{2} \sin \tfrac{1}{2}\text{BC} \sec \tfrac{1}{2}\text{AC}.$$

10. If c_1, c_2 be the two values of the third side when A, a, b are given and the triangle is ambiguous, shew that

$$\tan \tfrac{1}{2}c_1 \tan \tfrac{1}{2}c_2 = \tan \tfrac{1}{2}(b-a) \tan \tfrac{1}{2}(b+a).$$

EXAMPLES VII.

Solve the triangle in the following cases:

1. Given $c = 90°$, $a = 138° \, 4'$, $b = 109° \, 41'$.
Results. $C = 113° \, 28' \, 2''$, $A = 142° \, 11' \, 38''$, $B = 120° \, 15' \, 57''$.

2. Given $c = 90°$, $A = 131° \, 30'$, $B = 120° \, 32'$.
Results. $C = 109° \, 40' \, 20''$, $a = 127° \, 17' \, 51''$, $b = 113° \, 49' \, 31''$.

3. Given $a = 76° \, 35' \, 36''$, $b = 50° \, 10' \, 30''$, $c = 40° \, 0' \, 10''$.
Results. $A = 121° \, 36' \, 20''$, $B = 42° \, 15' \, 13''$, $C = 34° \, 15' \, 3''$.

4. Given $A = 129° \, 5' \, 28''$, $B = 142° \, 12' \, 42''$, $C = 105° \, 8' \, 10''$.
Results. $a = 135° \, 49' \, 20''$, $b = 144° \, 37' \, 15''$, $c = 60° \, 4' \, 54''$.

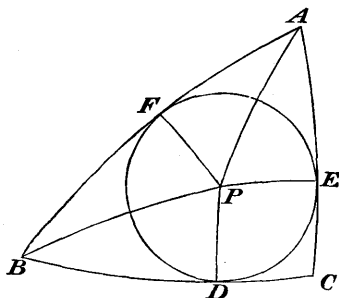
5. Solve the triangle having given two sides and the sum of the angles opposite to them.

6. Solve the triangle when the perimeter, the sum of two angles, and the third angle are given.

7. Solve the triangle when the perimeter and two angles are given.

CHAPTER VI.

CIRCUMSCRIBED AND INSCRIBED CIRCLES.

119. The inscribed circle. *To find the angular radius of the small circle inscribed in a given triangle.*

Let ABC be the triangle; bisect the angles A and B by arcs meeting at P; from P draw PD, PE, PF perpendicular to the sides.

Then the triangles PEA, PFA, having their angles at E and F right-angles, their angles at A equal, and the side PA common, are equal in all respects; in like manner the triangles PFB, PDB are equal; consequently $PE = PF = PD$.

And the triangles PCD, PCE, having their angles at D and E right-angles, the side PC common, and PD equal to PE, are equal in all respects; hence CP bisects the angle C.

Thus the arcs bisecting internally the angles of the triangle ABC pass through a common point P; and the arcs PD, PE, PF, drawn from P perpendicular to the sides, are all of the same length. This length we denote by r.

Accordingly the small circle whose pole is P and radius r touches the sides of the triangle at D, E, F, and is therefore called the *inscribed circle* of the triangle.

From the equalities of triangles already established it follows that $AE = AF$, $BF = BD$, $CD = CE$; hence $BC + AF =$ half the sum of the sides of the triangle $= s$; and $AF = s - a$.

Now $$\tan PF = \tan P\hat{A}F \sin AF \quad \text{(Art. 73)};$$

thus $$\tan r = \tan \tfrac{1}{2}A \sin(s-a). \quad \text{.....(1)}$$

The value of tan r may be expressed in various forms; thus, from Art. 50, we obtain

$$\tan\frac{A}{2} = \sqrt{\left\{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}\right\}};$$

substitute this value in (1), thus

$$\tan r = \sqrt{\left\{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}\right\}} = \frac{n}{\sin s} \quad \text{(Art. 51)}. \ \ldots(2)$$

Again $$\sin(s-a) = \frac{\sin a \sin \frac{1}{2}B \sin \frac{1}{2}C}{\sin \frac{1}{2}A}$$

by formula (11) of Art. 50, and the corresponding expressions for $\sin \frac{1}{2}B$ and $\sin \frac{1}{2}C$; therefore, from (1),

$$\tan r = \frac{\sin \frac{1}{2}B \sin \frac{1}{2}C}{\cos \frac{1}{2}A} \sin a; \quad \text{.....(3)}$$

hence, by Art. 58,

$$\tan r = \frac{\sqrt{\{-\cos S \cos(S-A)\cos(S-B)\cos(S-C)\}}}{2\cos\frac{1}{2}A \cos\frac{1}{2}B \cos\frac{1}{2}C}$$

$$= \frac{N}{2\cos\frac{1}{2}A \cos\frac{1}{2}B \cos\frac{1}{2}C}. \text{*} \quad \text{.....(4)}$$

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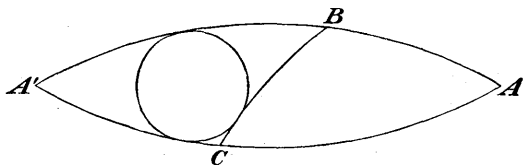
* LEXELL (*Acta Petropolitana*, 1782).

It may easily be shewn that

$$4\cos\tfrac{1}{2}\mathrm{A}\cos\tfrac{1}{2}\mathrm{B}\cos\tfrac{1}{2}\mathrm{C} = \cos\mathrm{S} + \cos(\mathrm{S}-\mathrm{A}) + \cos(\mathrm{S}-\mathrm{B}) + \cos(\mathrm{S}-\mathrm{C});$$

hence we have from (4)

$$\cot r = \frac{1}{2\mathrm{N}}\{\cos\mathrm{S} + \cos(\mathrm{S}-\mathrm{A}) + \cos(\mathrm{S}-\mathrm{B}) + \cos(\mathrm{S}-\mathrm{C})\}. \quad \ldots\ldots(5)$$

120. The escribed circles. *To find the angular radius of the small circle described so as to touch one side of a given triangle, and the other sides produced.*

Let ABC be the triangle; and suppose we require the radius of the small circle which touches BC, and AB and AC produced. Produce AB and AC to meet at A′; then we require the radius of the small circle *inscribed in* A′BC, and the sides of A′BC are a, $\pi - b$, $\pi - c$ respectively. Hence if r_1 be the required radius, and s denote as usual $\frac{1}{2}(a+b+c)$, we have from Art. 119,

$$\tan r_1 = \tan\frac{\mathrm{A}}{2}\sin s. \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

From this result we may derive other equivalent forms as in the preceding Article; or we may make use of those forms immediately, observing that the angles of the triangle A′BC are A, $\pi - \mathrm{B}$, $\pi - \mathrm{C}$ respectively. Hence, s being $\frac{1}{2}(a+b+c)$ and S being $\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})$, we shall obtain

$$\tan r_1 = \sqrt{\left\{\frac{\sin s\sin(s-b)\sin(s-c)}{\sin(s-a)}\right\}} = \frac{n}{\sin(s-a)}, \quad \ldots\ldots(7)$$

$$\tan r_1 = \frac{\cos\frac{1}{2}\mathrm{B}\cos\frac{1}{2}\mathrm{C}}{\cos\frac{1}{2}\mathrm{A}}\sin a, \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

$$\tan r_1 = \frac{\sqrt{\{-\cos S \cos(S-A)\cos(S-B)\cos(S-C)\}}}{2\cos\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C}$$

$$= \frac{N}{2\cos\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C}, \qquad\qquad (9)$$

$$\cot r_1 = \frac{1}{2N}\left\{-\cos S - \cos(S-A) + \cos(S-B) + \cos(S-C)\right\}. \qquad (10)$$

These results may also be found independently by bisecting two of the angles of the triangle A′BC, so as to determine the pole of the small circle, and proceeding as in Art. 119.

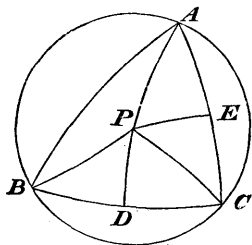
121. A circle which touches one side of a triangle and the other sides produced is called an *escribed circle*; thus there are three escribed circles belonging to a given triangle. We may denote the radii of the escribed circles which touch CA and AB respectively by r_2 and r_3, and values of $\tan r_2$ and $\tan r_3$ may be found, from what has already been given with respect to $\tan r_1$, by appropriate changes in the letters which denote the sides and angles.

In the preceding Article a triangle A′BC was formed by producing AB and AC to meet again at A′; similarly another triangle may be formed by producing BC and BA to meet again, and another by producing CA and CB to meet again. These are the *colunar* triangles of ABC. The original triangle ABC and the three formed from it have been called *associated triangles*, ABC being the fundamental triangle. Thus the inscribed and escribed circles of a given triangle are the same as the circles inscribed in the system of associated triangles of which the given triangle is the fundamental triangle.

122. The circum-circle. *To find the angular radius of the small circle described about a given triangle.*

Let ABC be the given triangle; bisect the sides CB, CA at D and E respectively, and draw from D and E arcs at right angles to CB and CA respectively, and let P be the intersection

of these arcs. Then a small circle, having its pole at P, can be described about ABC.

To prove this we draw the arcs PA, PB, PC, and observe that the triangles PEA, PEC, having their angles at E right-angles, the side PE common, and the sides EA, EC equal, are equal in all respects; in like manner the triangles PDB, PDC are equal; consequently $PA = PC = PB$.

And, if F be the mid point of AB, the triangles PFA, PFB have their sides equal each to each; therefore the angles PFA, PFB are equal to one another and so are right-angles.

Thus the arcs drawn through the mid points of the sides of the triangle ABC perpendicular to the sides respectively, pass through a common point P; and the arcs PA, PB, PC, joining P to the corners, are of the same length. This length we denote by R.

Accordingly the small circle whose pole is P and radius R passes through the corners of the triangle, and is therefore called the *circumscribed circle*, or *circum-circle*, of the triangle.

From the equalities of triangles already established it follows that $P\hat{B}C = P\hat{C}B$, $P\hat{C}A = P\hat{A}C$, $P\hat{A}B = P\hat{B}A$; hence $P\hat{C}B + A = \frac{1}{2}(A + B + C) = S$; and $P\hat{C}B = S - A$.

Now $$\tan CD = \tan CP \cos P\hat{C}D \quad \text{(Art. 73)},$$

thus $$\tan \tfrac{1}{2}a = \tan R \cos (S - A),$$

therefore $$\tan \mathsf{R} = \frac{\tan \frac{1}{2}a}{\cos (\mathsf{S} - \mathsf{A})}. \qquad \text{(11)}$$

The value of tan R may be expressed in various forms; thus if we substitute for $\tan \frac{1}{2}a$ from Art. 56, we obtain

$$\tan \mathsf{R} = \sqrt{\left\{\frac{-\cos \mathsf{S}}{\cos (\mathsf{S}-\mathsf{A}) \cos (\mathsf{S}-\mathsf{B}) \cos (\mathsf{S}-\mathsf{C})}\right\}} = -\frac{\cos \mathsf{S}}{\mathsf{N}}. \qquad \text{(12)}$$

Again $$\cos (\mathsf{S} - \mathsf{A}) = \frac{\sin \mathsf{A}}{\cos \frac{1}{2}a} \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c$$

by formula (29) of Art. 56, and the corresponding formulae for $\cos\frac{1}{2}b$ and $\cos\frac{1}{2}c$; therefore, from (11),

$$\tan \mathsf{R} = \frac{\sin \frac{1}{2}a}{\sin \mathsf{A} \cos \frac{1}{2}b \cos \frac{1}{2}c}. \qquad \text{(13)}$$

Substitute in the last expression the value of sin A from Art. 51; thus

$$\tan \mathsf{R} = \frac{2 \sin \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c}{\sqrt{\{\sin s \sin (s-a) \sin (s-b) \sin (s-c)\}}}$$

$$= \frac{2 \sin \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c}{n}. \qquad \text{(14)*}$$

It may easily be shewn that
$4 \sin \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c = \sin (s-a) + \sin (s-b) + \sin (s-c) - \sin s$; hence we have from (14)

$$\tan \mathsf{R} = \frac{1}{2n} \{\sin (s-a) + \sin (s-b) + \sin (s-c) - \sin s\}. \qquad \text{(15)}$$

123. *To find the angular radii of the small circles described round the triangles associated with a given fundamental triangle.*

Let R_1 denote the radius of the circle described round the triangle formed by producing AB and AC to meet again at A′; similarly let R_2 and R_3 denote the radii of the circles described round the other two triangles which are similarly formed.

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* Lexell, *loc. cit.* Cf. Gudermann, *Niedere Sphärik*, § 131.

Then we may deduce expressions for $\tan R_1$, $\tan R_2$, and $\tan R_3$ from those found in Art. 122 for $\tan R$. The sides of the triangle $A'BC$ are a, $\pi - b$, $\pi - c$, and its angles are A, $\pi - B$, $\pi - C$; hence if $s = \frac{1}{2}(a + b + c)$ and $S = \frac{1}{2}(A + B + C)$ we shall obtain from Art. 122:

$$\tan R_1 = \frac{\tan \frac{1}{2}a}{-\cos S}, \qquad (16)$$

$$\tan R_1 = \sqrt{\left\{\frac{\cos(S - A)}{-\cos S \cos(S - B)\cos(S - C)}\right\}} = \frac{\cos(S - A)}{N}, \qquad (17)$$

$$\tan R_1 = \frac{\sin \frac{1}{2}a}{\sin A \sin \frac{1}{2}b \sin \frac{1}{2}c}, \qquad (18)$$

$$\tan R_1 = \frac{2 \sin \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c}{\sqrt{\{\sin s \sin(s - a) \sin(s - b) \sin(s - c)\}}}, \qquad (19)$$

$$\tan R_1 = \frac{1}{2n}\{\sin s - \sin(s - a) + \sin(s - b) + \sin(s - c)\} \qquad (20)$$

Similarly we may find expressions for $\tan R_2$ and $\tan R_3$.

124. Applications of preceding formulae. Many examples may be proposed involving properties of the circles inscribed in and described about the associated triangles. We shall give one that will be of use hereafter.

To prove that

$$(\cot r + \tan R)^2 = \frac{1}{4n^2}(\sin a + \sin b + \sin c)^2 - 1. \qquad (21)$$

We have

$$4n^2 = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c; \qquad (22)$$

therefore

$$(\sin a + \sin b + \sin c)^2 - 4n^2$$

$$= 2(1 + \sin b \sin c + \sin c \sin a + \sin a \sin b - \cos a \cos b \cos c).$$

Also

$$\cot r + \tan R = \frac{1}{2n}\{\sin s + \sin(s - a) + \sin(s - b) + \sin(s - c)\}; \qquad (23)$$

and by squaring both members of this equation the required result will be obtained. For

$$\sin s + \sin(s-a) + \sin(s-b) + \sin(s-c)$$
$$= 2[\cos\tfrac{1}{2}a \sin\tfrac{1}{2}(b+c) + \sin\tfrac{1}{2}a \cos\tfrac{1}{2}(b-c)],$$

and the square of this

$$= (1+\cos a)\{1-\cos(b+c)\} + (1-\cos a)\{1+\cos(b-c)\} + 2\sin a\{\sin b + \sin c\}$$
$$= 2(1 + \sin b \sin c + \sin c \sin a + \sin a \sin b - \cos a \cos b \cos c).$$

Similarly we may prove that

$$(\cot r_1 - \tan R)^2 = \frac{1}{4n^2}(\sin b + \sin c - \sin a)^2 - 1 \ldots\ldots(24)$$

125. In the figure to Art. 119, suppose DP produced through P to a point A′ such that DA′ is a quadrant, then A′ is a pole of BC, and $PA' = \frac{1}{2}\pi - r$; similarly, suppose EP produced through P to a point B′ such that EB′ is a quadrant, and FP produced through P to a point C′ such that FC′ is a quadrant. Then A′B′C′ is the polar triangle of ABC, and $PA' = PB' = PC' = \frac{1}{2}\pi - r$. Thus P is the pole of the small circle *described round* the polar triangle, and the angular radius of the circum-circle of the polar triangle is the complement of the angular radius of the inscribed circle of the primitive triangle. And in like manner the point which is the pole of the small circle inscribed in the polar triangle is also the pole of the small circle described round the primitive triangle, and the angular radii of the two circles are complementary.

EXAMPLES VIII.

In the following examples the notation of the Chapter is retained.

Shew that in any triangle the relations contained in Examples 1 to 12 hold good.

1. $\text{Tan}\, r_1 \tan r_2 \tan r_3 = \tan r \sin^2 s$.

2. $\text{Tan}\, R + \cot r = \tan R_1 + \cot r_1 = \tan R_2 + \cot r_2 = \tan R_3 + \cot r_3$
$= \frac{1}{2}(\cot r + \cot r_1 + \cot r_2 + \cot r_3)$.

3. $\text{Tan}^2 R + \tan^2 R_1 + \tan^2 R_2 + \tan^2 R_3 = \cot^2 r + \cot^2 r_1 + \cot^2 r_2 + \cot^2 r_3$.

4. $\dfrac{\text{Tan}\, r_1 + \tan r_2 + \tan r_3 - \tan r}{\cot r_1 + \cot r_2 + \cot r_3 - \cot r} = \frac{1}{2}(1 + \cos a + \cos b + \cos c)$.

5. $\text{Cosec}^2 r = \cot(s-b)\cot(s-c) + \cot(s-c)\cot(s-a) + \cot(s-a)\cot(s-b)$.

6. $\text{Cosec}^2 r_1 = \cot(s-b)\cot(s-c) - \cot s \cot(s-b) - \cot s \cot(s-c)$.

7. $\text{Tan}\, R_1 \tan R_2 \tan R_3 = \tan R \sec^2 S$.

8. $\text{Tan}\, R = \frac{1}{2}(\cot r_1 + \cot r_2 + \cot r_3 - \cot r)$.

9. $\text{Cot}\, r = \frac{1}{2}(\tan R_1 + \tan R_2 + \tan R_3 - \tan R)$.

10. $\text{Tan}\, R \tan R_1 + \tan R_2 \tan R_3 = \cot r \cot r_1 + \cot r_2 \cot r_3$.

11. $\text{Tan}\, R \tan R_1 \tan R_2 \tan R_3 = \dfrac{4}{\sin^2 a \sin^2 B \sin^2 C} = \dfrac{1}{N^2}$.

12. $\text{Cot}\, r \cot r_1 \cot r_2 \cot r_3 = \dfrac{4}{\sin^2 b \sin^2 c \sin^2 A} = \dfrac{1}{n^2}$.

13. Shew that in an equilateral triangle $\tan R = 2 \tan r$.

14. If ABC be an equilateral spherical triangle, P the pole of the circle circumscribing it, Q any point on the sphere, shew that

$$\cos QA + \cos QB + \cos QC = 3 \cos PA \cos PQ.$$

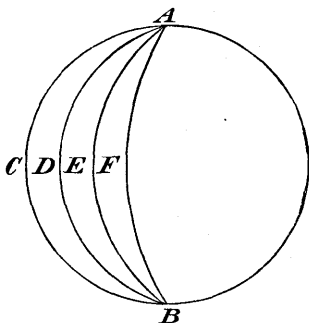
15. If three small circles be inscribed in a spherical triangle having each of its angles 120°, so that each touches the other two as well as two sides of the triangle, shew that the radius of each of the small circles $= 30°$, and that the centres of the three small circles coincide with the angular points of the polar triangle.

16. Solve the spherical triangle when the radius of the inscribed circle and two angles are given.

17. Solve the spherical triangle when the radius of the circumscribed circle and two sides are given.

CHAPTER VII.

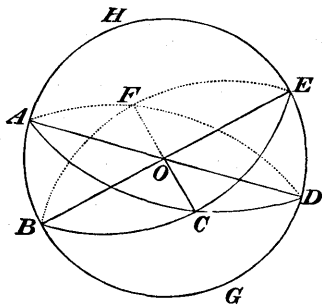
AREA OF A SPHERICAL TRIANGLE. SPHERICAL EXCESS.

126. *To find the area of a Lune.*

Let $ACBDA$, $ADBEA$ be two lunes having equal angles at A; then one of these lunes may be supposed placed on the other so as to coincide exactly with it; thus *lunes having equal angles are equal.* Then, by a process similar to that used in the first proposition of the Sixth Book of Euclid, it may be shewn that *lunes are proportional to their angles.* Hence, since the whole surface of a sphere may be considered as a lune with an angle equal to four right angles, we have for a lune with an angle whose circular measure is A,

$$\frac{\text{area of lune}}{\text{surface of sphere}} = \frac{A}{2\pi}.$$

Suppose r the radius of the sphere, then the surface is $4\pi r^2$ (*Integral Calculus*, Chap. VII.); thus

$$\text{area of lune} = \frac{\mathsf{A}}{2\pi} 4\pi r^2 = 2\mathsf{A}r^2. \qquad \text{.................(1)}$$

127. Girard's theorem.* *To find the area of a Spherical Triangle.*

Let ABC be a spherical triangle; produce the arcs which form its sides until they meet again two and two, which will happen when each has become equal to the semi-circumference. The triangle ABC now forms a part of three lunes, namely, ABDCA, BCEAB, and CAFBC. Now the triangles CDE and FAB are subtended by vertically opposite solid angles at O, and we shall assume that their areas are equal; therefore the lune CAFBC is equal to the sum of the two triangles ABC and CDE. Hence, if A, B, C denote the circular measures of the angles of the triangle, we have

$$\text{triangle ABC} + \text{BGDC} = \text{lune ABDCA} = 2\mathsf{A}r^2,$$
$$\text{triangle ABC} + \text{AHEC} = \text{lune BCEAB} = 2\mathsf{B}r^2,$$
$$\text{triangle ABC} + \text{triangle CDE} = \text{lune CAFBC} = 2\mathsf{C}r^2;$$

* This theorem was published by GIRARD in his *Invention nouvelle en Algèbre*, printed at Amsterdam in 1629. His proof, however, is not rigorous, so the theorem may be attributed to CAVALIERI, who gave a strict proof in his *Directorium generale uraniometricum*, printed at Bologna in 1632.

hence, by addition,

twice triangle ABC + surface of hemisphere $= 2(A + B + C)r^2$;

therefore triangle $ABC = (A + B + C - \pi)r^2$................(2)

Spherical excess. The expression $A + B + C - \pi$ is called the *spherical excess* of the triangle; and since

$$(A + B + C - \pi)r^2 = \frac{A + B + C - \pi}{4\pi} 4\pi r^2,$$

the result obtained may be enunciated thus: *the area of a spherical triangle is the same fraction of the surface of the sphere as the spherical excess is of eight right angles.*

128. We have assumed, as is usually done, that the areas of the triangles CDE and FAB in the preceding Article are equal. The triangles, however, are not *congruent*, but are *symmetrically* equal (Art. 33), so that one cannot be made to coincide with the other by superposition. It is, however, easy to decompose two such triangles into pieces which admit of superposition, and thus to prove that their areas are equal. For describe a small circle round each; then, since the triangles have the same elements, the angular radii of these circles will be equal by Art. 122. If the pole of the circumscribing circle falls inside each triangle, each triangle is the sum of three isosceles triangles, and if the pole falls outside each triangle, each triangle is the excess of two isosceles triangles over a third; and in each case the isosceles triangles of one set are respectively *absolutely equal* to the corresponding isosceles triangles of the other set, since in the case of isosceles triangles the distinction between *congruence* and *symmetrical equality* does not exist.

129. *To find the area of a spherical polygon.*

Let n be the number of sides of the polygon, Σ the sum of all its angles. Take any point within the polygon and join it

with all the angular points; thus the figure is divided into n triangles. Hence, by Art. 127,

$$\text{area of polygon} = (\text{sum of the angles of the triangles} - n\pi)r^2,$$

and the sum of the angles of the triangles is equal to Σ together with the four right angles which are formed round the common vertex; therefore

$$\text{area of polygon} = \{\Sigma - (n-2)\pi\}r^2 \text{..............} (3)$$

This expression is true even when the polygon has some of its angles greater than two right angles, provided it can be decomposed into triangles each of whose angles is less than two right angles.

130. Definition. The *spherical excess* of a spherical polygon is the excess of the sum of the angles of the polygon over the sum of the angles of a plane polygon having the same number of sides.

The result just obtained shews that the area of a spherical polygon is proportional to its spherical excess.*

131. We shall now give some expressions for certain trigonometrical functions of the *spherical excess* of a triangle. We denote the spherical excess by E, so that $\mathsf{E} = \mathsf{A} + \mathsf{B} + \mathsf{C} - \pi$; and it is most important to notice that, with the notation of Art. 56,

$$\mathsf{S} = \tfrac{1}{2}\mathsf{E} + \tfrac{1}{2}\pi \text{............................} (4)$$

A geometrical representation of the spherical excess is given in the next chapter, Art. 149.

132. Cagnoli's theorem.† *To shew that*

$$\sin \tfrac{1}{2}\mathsf{E} = \frac{\sqrt{\{\sin s \sin (s-a) \sin (s-b) \sin (s-c)\}}}{2 \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c}.$$

* Cf. Gauss, *Disq. circa Superficies Curvas*, § 20.

† Cagnoli, *Trigonometria*, § 1146. See also Lexell, *Acta Petrop.*, 1782, p. 68.

Making use of the expressions for the sines and cosines of the half sides obtained in Art. 56, we see that

$$\frac{\sin \mathsf{C} \sin \frac{1}{2}a \sin \frac{1}{2}b}{\cos \frac{1}{2}c} = \sqrt{\cos^2 \mathsf{S}} = -\cos \mathsf{S} = \sin \tfrac{1}{2}\mathsf{E}, \quad \text{.........(5)}$$

the sign of the square root being determined from the consideration that all the factors of the left-hand expression are necessarily positive, while $\cos \mathsf{S}$ is essentially negative, since S is greater than $\frac{1}{2}\pi$, and less than $\frac{3}{2}\pi$ (Art. 32).

Now,

$$\sin \mathsf{C} = \frac{2n}{\sin a \sin b} = \frac{n}{2 \sin \frac{1}{2}a \cos \frac{1}{2}a \sin \frac{1}{2}b \cos \frac{1}{2}b}, \quad \text{(Art. 51)},$$

and therefore

$$\sin \tfrac{1}{2}\mathsf{E} = \frac{n}{2 \cos \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c}, \quad \text{..................(6)}$$

as was to be proved.

In the same way it is seen that

$$\frac{\sin \mathsf{C} \cos \frac{1}{2}a \cos \frac{1}{2}b}{\cos \frac{1}{2}c} = \cos (\mathsf{S} - \mathsf{C}) = \sin (\mathsf{C} - \tfrac{1}{2}\mathsf{E}),$$

the last sine being assumed essentially positive; whence

$$\sin (\mathsf{C} - \tfrac{1}{2}\mathsf{E}) = \frac{n}{2 \sin \frac{1}{2}a \sin \frac{1}{2}b \cos \frac{1}{2}c}. \quad \text{................(7)}$$

133. A better method of obtaining this formula for $\sin(\mathsf{C} - \frac{1}{2}\mathsf{E})$ is to apply CAGNOLI'S theorem to the colunar triangle $\mathsf{ABC'}$. If we employ accented letters to denote the elements of this triangle, we have the relations

$$\left.\begin{array}{lll} \mathsf{A}' = \pi - \mathsf{A}, & \mathsf{B}' = \pi - \mathsf{B}, & \mathsf{C}' = \mathsf{C}, \\ a' = \pi - a, & b' = \pi - b, & c' = c, \end{array}\right\} \quad \text{..............(8)}$$

whence also

$$\left.\begin{array}{c} s' = \pi - (s - c), \quad s' - a' = s - b, \quad s' - b' = s - a, \quad s' - c' = \pi - s, \\ n' = n, \\ \mathsf{E}' = 2\mathsf{C} - \mathsf{E}\,; \end{array}\right\} \quad \text{....(9)}$$

the substitution of these values in CAGNOLI'S formula yields result (7) of the previous Article.

It should be noticed that E', being a spherical excess, is necessarily positive and less than 2π; hence $C - \frac{1}{2}E$ is positive and less than π. This justifies the assumption made above, that $\sin(C - \frac{1}{2}E)$ is always positive.

134. L'Huilier's theorem.* *To shew that*

$$\tan\tfrac{1}{4}E = \surd\{\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)\}.$$

$$\text{Tan}\,\tfrac{1}{4}E = \frac{\sin\frac{1}{4}(A+B+C-\pi)}{\cos\frac{1}{4}(A+B+C-\pi)}$$

$$= \frac{\sin\frac{1}{2}(A+B) - \sin\frac{1}{2}(\pi - C)}{\cos\frac{1}{2}(A+B) + \cos\frac{1}{2}(\pi - C)},$$

$$= \frac{\sin\frac{1}{2}(A+B) - \cos\frac{1}{2}C}{\cos\frac{1}{2}(A+B) + \sin\frac{1}{2}C}$$

$$= \frac{\cos\frac{1}{2}(a-b) - \cos\frac{1}{2}c}{\cos\frac{1}{2}(a+b) + \cos\frac{1}{2}c}\cdot\frac{\cos\frac{1}{2}C}{\sin\frac{1}{2}C}. \qquad \text{(Art. 63).}$$

Hence, by Art. 50, we obtain

$$\tan\tfrac{1}{4}E = \frac{\sin\frac{1}{4}(c+a-b)\sin\frac{1}{4}(c+b-a)}{\cos\frac{1}{4}(a+b+c)\cos\frac{1}{4}(a+b-c)}\sqrt{\left\{\frac{\sin s\sin(s-c)}{\sin(s-a)\sin(s-b)}\right\}}$$

$$= \surd\{\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)\}. \quad \text{....(10)}$$

135. Gent's† proof of L'Huilier's theorem. The first and third of DELAMBRE'S analogies (Art. 63), may be written in the form

$$\frac{\cos\frac{1}{2}(C-E)}{\cos\frac{1}{2}C} = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}, \quad \text{................(11)}$$

$$\frac{\sin\frac{1}{2}(C-E)}{\sin\frac{1}{2}C} = \frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c}. \quad \text{................(12)}$$

From (11) we deduce

$$\frac{\cos\frac{1}{2}(C-E) - \cos\frac{1}{2}C}{\cos\frac{1}{2}(C-E) + \cos\frac{1}{2}C} = \frac{\cos\frac{1}{2}(a-b) - \cos\frac{1}{2}c}{\cos\frac{1}{2}(a-b) + \cos\frac{1}{2}c},$$

which is the same as

$$\tan\tfrac{1}{4}E\tan\tfrac{1}{4}(2C - E) = \tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b). \quad \text{...(13)}$$

* L'HUILIER (LEGENDRE, *Géométrie*, Note 10).

† GRUNERT'S *Archiv der Math. und Physik*, XX, 1853, p. 358.

Treating equation (12) in a similar manner, we get

$$\tan \tfrac{1}{4}\mathsf{E} \cot \tfrac{1}{4}(2\mathsf{C} - \mathsf{E}) = \tan \tfrac{1}{2}s \tan \tfrac{1}{2}(s - c). \quad \text{........(14)}$$

Now multiply corresponding sides of (13) and (14), take the square root, and the result is L'HUILIER'S formula

$$\tan \tfrac{1}{4}\mathsf{E} = \sqrt{\tan \tfrac{1}{2}s \tan \tfrac{1}{2}(s - a) \tan \tfrac{1}{2}(s - b) \tan \tfrac{1}{2}(s - c)}. \quad \text{...(15)}$$

Also the division of (13) by (14) gives

$$\tan \tfrac{1}{4}(2\mathsf{C} - \mathsf{E}) = \sqrt{\cot \tfrac{1}{2}s \tan \tfrac{1}{2}(s - a) \tan \tfrac{1}{2}(s - b) \cot \tfrac{1}{2}(s - c)}. \quad \text{..(16)}$$

136. If we apply L'HUILIER'S theorem to the polar triangle, whose spherical excess is $2(\pi - s)$,* and semiperimeter $\pi - \frac{1}{2}\mathsf{E}$, we get

$$\cot \tfrac{1}{2}s = \sqrt{\cot \tfrac{1}{4}\mathsf{E} \tan \tfrac{1}{4}(2\mathsf{A} - \mathsf{E}) \tan \tfrac{1}{4}(2\mathsf{B} - \mathsf{E}) \tan \tfrac{1}{4}(2\mathsf{C} - \mathsf{E})}; \quad (17)$$

while corresponding to (16) we have

$$\tan \tfrac{1}{2}(s - c) = \sqrt{\tan \tfrac{1}{4}\mathsf{E} \tan \tfrac{1}{4}(2\mathsf{A} - \mathsf{E}) \tan \tfrac{1}{4}(2\mathsf{B} - \mathsf{E}) \cot \tfrac{1}{4}(2\mathsf{C} - \mathsf{E})}. \quad (18)$$

These results may also be obtained directly from the third and fourth of DELAMBRE'S analogies, by a method analogous to that used in the preceding Article.

137. The Lhuilierian. If we put

$$\mathsf{L} = \sqrt{\cot \tfrac{1}{2}s \tan \tfrac{1}{2}(s - a) \tan \tfrac{1}{2}(s - b) \tan \tfrac{1}{2}(s - c)}, \quad \text{......(19)}$$

equation (15) of Art. 135 may be written

$$\tan \tfrac{1}{4}\mathsf{E} = \frac{\mathsf{L}}{\cot \tfrac{1}{2}s}, \quad \text{......................(20)}$$

and equation (16) of Art. 135 may be written

$$\tan \tfrac{1}{4}(2\mathsf{C} - \mathsf{E}) = \frac{\mathsf{L}}{\tan \tfrac{1}{2}(s - c)}; \quad \text{...... .. ..(21)}$$

similarly

$$\tan \tfrac{1}{4}(2\mathsf{A} - \mathsf{E}) = \frac{\mathsf{L}}{\tan \tfrac{1}{2}(s - a)}, \quad \text{..............(22)}$$

and

$$\tan \tfrac{1}{4}(2\mathsf{B} - \mathsf{E}) = \frac{\mathsf{L}}{\tan \tfrac{1}{2}(s - b)}. \quad \text{..............(23)}$$

* The relation between the area of a figure and the perimeter of the reciprocal figure is referred to by SCHULZ, *Sphärik*, II, p. 241.

Now multiply together equations (20) to (23), and we get

$$\mathsf{L}=\sqrt{\tan\tfrac{1}{4}\mathsf{E}\tan\tfrac{1}{4}(2\mathsf{A}-\mathsf{E})\tan\tfrac{1}{4}(2\mathsf{B}-\mathsf{E})\tan\tfrac{1}{4}(2\mathsf{C}-\mathsf{E})}. \quad (24)$$

Dr. CASEY suggests that the function L should be called the *Lhuilierian* of the triangle. He also points out that, on account of its double value (formulae 19 and 24), it will give the solution of a spherical triangle when either the three sides or the three angles are given, the equations used in either case being (20), (21), (22), and (23).

138. Prouhet's* proof of Cagnoli's theorem. Equations (11) and (12) of Art. 135 may be written in the form

$$\cos\tfrac{1}{2}\mathsf{C}\cos\tfrac{1}{2}\mathsf{E}+\sin\tfrac{1}{2}\mathsf{C}\sin\tfrac{1}{2}\mathsf{E}=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}\cos\tfrac{1}{2}\mathsf{C},$$

$$\sin\tfrac{1}{2}\mathsf{C}\cos\tfrac{1}{2}\mathsf{E}-\cos\tfrac{1}{2}\mathsf{C}\sin\tfrac{1}{2}\mathsf{E}=\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c}\sin\tfrac{1}{2}\mathsf{C}.$$

Solving these for $\sin\frac{1}{2}\mathsf{E}$ and $\cos\frac{1}{2}\mathsf{E}$, we get

$$\sin\tfrac{1}{2}\mathsf{E}=\sin\tfrac{1}{2}\mathsf{C}\cos\tfrac{1}{2}\mathsf{C}\sec\tfrac{1}{2}c\{\cos\tfrac{1}{2}(a-b)-\cos\tfrac{1}{2}(a+b)\}$$

$$=\sin\mathsf{C}\sin\tfrac{1}{2}a\sin\tfrac{1}{2}b\sec\tfrac{1}{2}c \quad \ldots\ldots\ldots\ldots (25)$$

$$=\frac{n}{2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c} \quad \text{as in Art. 132.}$$

Also

$$\cos\tfrac{1}{2}\mathsf{E}=\{\cos\tfrac{1}{2}(a+b)\sin^2\tfrac{1}{2}\mathsf{C}+\cos\tfrac{1}{2}(a-b)\cos^2\tfrac{1}{2}\mathsf{C}\}\sec\tfrac{1}{2}c$$

$$=\{\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b+\sin\tfrac{1}{2}a\sin\tfrac{1}{2}b\cos\mathsf{C}\}\sec\tfrac{1}{2}c \quad \ldots\ldots (26)\dagger$$

$$=\frac{(1+\cos a)(1+\cos b)+\sin a\sin b\cos\mathsf{C}}{4\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c}$$

$$=\frac{1+\cos a+\cos b+\cos c}{4\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c} \quad \ldots\ldots\ldots\ldots (27)\ddagger$$

$$=\frac{\cos^2\frac{1}{2}a+\cos^2\frac{1}{2}b+\cos^2\frac{1}{2}c-1}{2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c} \quad \ldots\ldots\ldots\ldots (28)$$

* *Nouvelles Annales de Mathématiques*, 1st series, XV, 1856, p. 91.

† LAGRANGE, *Journal de l'École Polytechnique*, Cahier 6; LEGENDRE, *Géométrie*, Note 10. For a geometrical proof see Art. 150 below, also GUDERMANN, *Niedere Sphärik*, § 152.

‡ EULER, *Acta Petropolitana*, 1778.

From (25) and (26) we get by division

$$\tan \tfrac{1}{2}E = \frac{\sin \frac{1}{2}a \sin \frac{1}{2}b \sin C}{\cos \frac{1}{2}a \cos \frac{1}{2}b + \sin \frac{1}{2}a \sin \frac{1}{2}b \cos C}, \qquad \text{.................(29)}$$

and from (25) and (27)

$$\tan \tfrac{1}{2}E = \frac{2n}{1 + \cos a + \cos b + \cos c}. \qquad \text{.......................(30)*}$$

139. Other formulae. We may obtain other formulae involving trigonometrical functions of the spherical excess. For example:

To find $\sin \frac{1}{4}E$ *and* $\cos \frac{1}{4}E$.

In (28) put $1 - 2\sin^2\frac{1}{4}E$ for $\cos E$; thus

$$\sin^2\tfrac{1}{4}E = \frac{1 + 2\cos\frac{1}{2}a \cos\frac{1}{2}b \cos\frac{1}{2}c - \cos^2\frac{1}{2}a - \cos^2\frac{1}{2}b - \cos^2\frac{1}{2}c}{4\cos\frac{1}{2}a \cos\frac{1}{2}b \cos\frac{1}{2}c}.$$

By ordinary development we can shew that the numerator of the above fraction is equal to

$$4\sin\tfrac{1}{2}s \sin\tfrac{1}{2}(s-a) \sin\tfrac{1}{2}(s-b) \sin\tfrac{1}{2}(s-c);$$

therefore

$$\sin^2\tfrac{1}{4}E = \frac{\sin\frac{1}{2}s \sin\frac{1}{2}(s-a) \sin\frac{1}{2}(s-b) \sin\frac{1}{2}(s-c)}{\cos\frac{1}{2}a \cos\frac{1}{2}b \cos\frac{1}{2}c}. \quad \text{...(31)}$$

Similarly

$$\cos^2\tfrac{1}{4}E = \frac{\cos\frac{1}{2}s \cos\frac{1}{2}(s-a) \cos\frac{1}{2}(s-b) \cos\frac{1}{2}(s-c)}{\cos\frac{1}{2}a \cos\frac{1}{2}b \cos\frac{1}{2}c}. \quad \text{...(32)}$$

From these by division we get L'HUILIER'S theorem.

140. *To find* $\cos(C - \frac{1}{2}E)$.

Substitute in (28) the elements of the colunar triangle as given in (8) and (9). We get

$$\cos(C - \tfrac{1}{2}E) = \frac{\sin^2\frac{1}{2}a + \sin^2\frac{1}{2}b + \cos^2\frac{1}{2}c - 1}{2\sin\frac{1}{2}a \sin\frac{1}{2}b \cos\frac{1}{2}c}.$$

$$= \frac{\cos^2\frac{1}{2}c - \cos^2\frac{1}{2}a - \cos^2\frac{1}{2}b + 1}{2\sin\frac{1}{2}a \sin\frac{1}{2}b \cos\frac{1}{2}c}. \quad \text{.........(33)}$$

* DE GUA (*Mémoires de l Académie des Sciences*, Paris, 1783).

From this, or by substituting the elements of the colunar triangle in (31) and (32), we also get

$$\sin^2(\tfrac{1}{2}C - \tfrac{1}{4}E) = \frac{\cos\frac{1}{2}s \sin\frac{1}{2}(s-a) \sin\frac{1}{2}(s-b) \cos\frac{1}{2}(s-c)}{\sin\frac{1}{2}a \sin\frac{1}{2}b \cos\frac{1}{2}c}, \quad \text{.....(34)}$$

$$\cos^2(\tfrac{1}{2}C - \tfrac{1}{4}E) = \frac{\sin\frac{1}{2}s \cos\frac{1}{2}(s-a) \cos\frac{1}{2}(s-b) \sin\frac{1}{2}(s-c)}{\sin\frac{1}{2}a \sin\frac{1}{2}b \cos\frac{1}{2}c} \quad \text{.......(35)}$$

Again from (6) and (7) we get, by division, the formula

$$\cot\tfrac{1}{2}E = \cot C + \frac{\cot\frac{1}{2}a \cot\frac{1}{2}b}{\sin C}, \quad \text{...............(36)}$$

which is another form of (29).

141. Calculation of the spherical excess. When the numerical values of three of the elements of a spherical triangle are given, the spherical excess, and therefore also the area, may be calculated *directly* from the data in three cases, *viz.*

(1) When the three angles are given.

(2) When the three sides are given; use is then made of L'HUILIER'S formula.

(3) When two sides and the included angle are given. We then use formula (29), and may adapt it to logarithms by introducing the subsidiary angle ϕ such that

$$\tan\phi = \tan\tfrac{1}{2}a \cos C; \quad \text{.....................(37)}$$

the formula then becomes

$$\tan\tfrac{1}{2}E = \frac{\sin\frac{1}{2}b \tan C \sin\phi}{\cos(\phi - \frac{1}{2}b)}. \quad \text{................(38)}$$

EXAMPLES IX.

1. Find the angles and sides of an equilateral triangle whose area is one-fourth of that of the sphere on which it is described.

2. Find the surface of an equilateral and equiangular spherical polygon of n sides, and determine the value of each of the angles when the surface equals half the surface of the sphere.

3. If $a=b=\frac{1}{3}\pi$, and $c=\frac{1}{2}\pi$, shew that $E=\cos^{-1}\frac{7}{9}$.

4. If the angle C of a spherical triangle be a right angle, shew that

$$\sin\tfrac{1}{2}E=\sin\tfrac{1}{2}a\sin\tfrac{1}{2}b\sec\tfrac{1}{2}c, \quad \cos\tfrac{1}{2}E=\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\sec\tfrac{1}{2}c.$$

5. If the angle C be a right angle, shew that

$$\frac{\sin^2 c}{\cos c}\cos E=\frac{\sin^2 a}{\cos a}+\frac{\sin^2 b}{\cos b}.$$

6. If $a=b$ and $C=\frac{\pi}{2}$, shew that $\tan E=\frac{\sin^2 a}{2\cos a}$.

7. The sum of the angles in a right-angled triangle is less than four right angles.

8. Draw through a given point in the side of a spherical triangle an arc of a great circle cutting off a given part of the triangle.

9. In a spherical triangle if $\cos C=-\tan\frac{1}{2}a\tan\frac{1}{2}b$, then $C=A+B$.

10. If the angles of a spherical triangle be together equal to four right angles

$$\cos^2\tfrac{1}{2}a+\cos^2\tfrac{1}{2}b+\cos^2\tfrac{1}{2}c=1.$$

11. If r_1, r_2, r_3 be the radii of three small circles of a sphere of radius r which touch one another at P, Q, R, and A, B, C be the angles of the spherical triangle formed by joining their centres,

$$\text{area PQR}=(A\cos r_1+B\cos r_2+C\cos r_3-\pi)r^2.$$

12. Shew that

$$\sin s=\frac{\{\sin\frac{1}{2}E\sin(A-\frac{1}{2}E)\sin(B-\frac{1}{2}E)\sin(C-\frac{1}{2}E)\}^{\frac{1}{2}}}{2\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C}.$$

13. Find the area of a regular polygon of a given number of sides formed by arcs of great circles on the surface of a sphere; and hence deduce that, if α be the angular radius of a small circle, its area is to that of the whole surface of the sphere as $(1-\cos\alpha)$ is to 2.

14. A, B, C are the angular points of a spherical triangle; A′, B′, C′ are the middle points of the respectively opposite sides. If E be the spherical excess of the triangle, shew that

$$\cos\tfrac{1}{2}E=\frac{\cos A'B'}{\cos\frac{1}{2}c}=\frac{\cos B'C'}{\cos\frac{1}{2}a}=\frac{\cos C'A'}{\cos\frac{1}{2}b}.$$

15. If one of the arcs of great circles which join the middle points of the sides of a spherical triangle be a quadrant, shew that the other two are also quadrants.

EXAMPLES X.

1. If σ be the spherical excess of the polar triangle, and E', E'', E''' those of the colunar triangles,

$$\tan \tfrac{1}{4}\sigma = \sqrt{\tan \tfrac{1}{4}E' \tan \tfrac{1}{4}E'' \tan \tfrac{1}{4}E''' \cot \tfrac{1}{4}E}.$$

(PROUHET.)

2. In an equilateral triangle

$$\tan \tfrac{1}{4}E = \tan \tfrac{1}{4}a\sqrt{\tan \tfrac{3}{4}a \tan \tfrac{1}{4}a}.$$

3. If a and b are the equal sides of an isosceles triangle

$$\tan \tfrac{1}{4}E = \tan \tfrac{1}{4}c\sqrt{\tan \tfrac{1}{2}(a+\tfrac{1}{2}c) \tan \tfrac{1}{2}(a-\tfrac{1}{2}c)}.$$

4. Shew that

$$\tan \tfrac{1}{4}E \cot \tfrac{1}{2}(A - \tfrac{1}{2}E) = \tan \tfrac{1}{2}s \tan \tfrac{1}{2}(s-a).$$

5. Solve the spherical triangle when the sum of two sides, the third side, and the spherical excess are given.

6. The area of an isosceles right-angled spherical triangle is $\frac{1}{12}$ of the surface of the sphere: calculate the hypotenuse. (R. U. I., 1898.)

7. Find the angle of an equilateral spherical triangle covering $\frac{1}{20}$ of the surface of the sphere.

Shew also that two of its medians and one of its sides are together equal to half the circumference of a great circle. (R. U. I., 1895.)

8. A spherical triangle is such that the centre of its circum-circle is on the base, and the vertical angle is $\frac{3}{4}\pi$. If the radius of the sphere be one foot, find the area of the triangle in square feet.

(R. U. I., 1895.)

9. A regular spherical quadrilateral has each of its angles equal to 100°; calculate the ratio of its area to that of the sphere.

10. The area of a regular spherical polygon of n sides is one p^{th} part of the sphere; find its sides and angles. (R. U. I., 1893.)

11. A given lune is divided into two isosceles triangles, and the area of one of them is n times the area of the other; shew that

$$\tan \tfrac{1}{2}A \cos\theta = \tan\left(\frac{n-1}{n+1}\right)\frac{A}{2},$$

where A denotes the angle of the lune, and θ one of the equal sides.

Find the ultimate value of $\cos\theta$ when A becomes indefinitely small; and hence shew that the surface of a segment of a sphere is to the surface of the sphere as the height of the segment is to the diameter of the sphere. (Sci. and Art, 1894.)

12. The side AB of a spherical triangle ABC is bisected at D. If E_1 and E_2 be the spherical excesses of the triangles ACD and BCD respectively, shew that

$$\sin\tfrac{1}{2}E_1\cos\tfrac{1}{2}b=\sin\tfrac{1}{2}E_2\cos\tfrac{1}{2}a.$$

(Sci. and Art, 1897.)

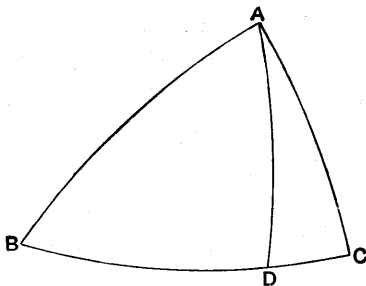
13. If a, b, c, d are the sides, $2p$ the perimeter, and 2S the area of a spherical quadrilateral inscribed in a small circle on a sphere of unit radius, shew that

$$\sin^2\tfrac{1}{2}S=\frac{\sin\frac{1}{2}(p-a)\sin\frac{1}{2}(p-b)\sin\frac{1}{2}(p-c)\sin\frac{1}{2}(p-d)}{\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c\cos\frac{1}{2}d}.$$

(Sci. and Art, 1898.)

CHAPTER VIII.

VARIOUS PROPERTIES OF THE SPHERICAL TRIANGLE.

142. Definition of altitude. The arc AD, drawn from a corner A to intersect the opposite side orthogonally in D, is called an *altitude* of the triangle.

Properties of the altitudes.

The product of the sine of a side and the sine of the corresponding altitude has the same value, whichever side be taken. This value may be called $2n$.

The product of the sine of an angle and the sine of the corresponding altitude has the same value, whichever angle be taken. This value may be called 2N.

To prove these propositions, we notice that in the right-angled triangles ADB, ADC

$$\sin c \sin \mathrm{B} = \sin \mathrm{AD} = \sin b \sin \mathrm{C}. \quad \text{.................}(1)$$

Hence

$$\sin a \sin \text{AD} = \sin c \sin a \sin \text{B} = \sin a \sin b \sin \text{C}$$
$$= \sin b \sin c \sin \text{A} = 2n, \quad \ldots\ldots(2)$$

$$\sin \text{A} \sin \text{AD} = \sin b \sin \text{C} \sin \text{A} = \sin c \sin \text{A} \sin \text{B}$$
$$= \sin a \sin \text{B} \sin \text{C} = 2\text{N}. \quad \ldots\ldots(3)$$

From these we derive the following

$$\frac{\sin a \sin b \sin c}{2n} = \frac{\sin a}{\sin \text{A}} = \frac{\sin b}{\sin \text{B}} = \frac{\sin c}{\sin \text{C}} = \frac{n}{\text{N}}; \quad \ldots\ldots\ldots(4)^*$$

$$\frac{\sin \text{A} \sin \text{B} \sin \text{C}}{2\text{N}} = \frac{\sin \text{A}}{\sin a} = \frac{\sin \text{B}}{\sin b} = \frac{\sin \text{C}}{\sin c} = \frac{\text{N}}{n}; \quad \ldots\ldots\ldots(5)^*$$

$$\text{N} = \frac{2n^2}{\sin a \sin b \sin c}, \quad n = \frac{2\text{N}^2}{\sin \text{A} \sin \text{B} \sin \text{C}}. \quad \ldots\ldots\ldots\ldots(6)$$

If we eliminate A from the equations

$$2n = \sin b \sin c \sin \text{A},$$
$$\cos a - \cos b \cos c = \sin b \sin c \cos \text{A},$$

namely by squaring and adding, we get

$$4n^2 = \sin^2 b \sin^2 c - (\cos a - \cos b \cos c)^2$$
$$= 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c. \quad \ldots\ldots(7)$$
$$= 4 \sin s \sin(s-a) \sin(s-b) \sin(s-c). \quad \ldots\ldots\ldots\ldots\ldots(8)$$

Similarly

$$4\text{N}^2 = \sin^2 \text{B} \sin^2 \text{C} - (\cos \text{A} + \cos \text{B} \cos \text{C})^2$$
$$= 1 - \cos^2 \text{A} - \cos^2 \text{B} - \cos^2 \text{C} - 2 \cos \text{A} \cos \text{B} \cos \text{C} \quad \ldots(9)$$
$$= -4 \cos \text{S} \cos(\text{S}-\text{A}) \cos(\text{S}-\text{B}) \cos(\text{S}-\text{C}). \quad \ldots\ldots(10)$$

Thus it is seen that the n and N of the present Article are the same as those defined in Arts. 51 and 58.

143. Length of any line drawn from a corner to the opposite side.

Let F be any point in the side BC, and let AF be joined.

* Cf. GUDERMANN, *Niedere Sphärik*, § 142.

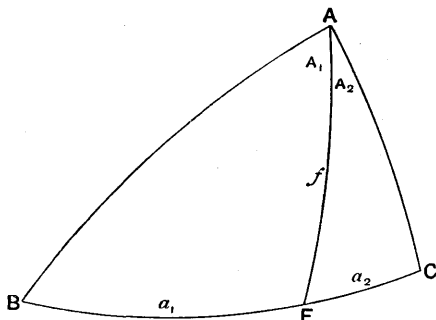
Let the lengths of the arcs AF, BF, FC be denoted by f, a_1, a_2 respectively, and the angles BAF, FAC by A_1, A_2 respectively.

Applying the cosine formula to the triangles BFA, CFA, we get

$$\frac{\cos c - \cos a_1 \cos f}{\sin a_1 \sin f} = \cos \hat{AFB}$$

$$= -\cos \hat{AFC} = \frac{\cos a_2 \cos f - \cos b}{\sin a_2 \sin f},$$

which gives

$$\cos f = \cos b \frac{\sin a_1}{\sin a} + \cos c \frac{\sin a_2}{\sin a} \quad \text{.................(11)}$$

Divide both sides by $\sin f$, and then multiply numerator and denominator of the right-hand side by $\sin F$; thus

$$\cot f = \frac{\cos b \sin a_1 \sin F + \cos c \sin a_2 \sin F}{\sin a \sin f \sin F}.$$

But

$$\sin a_1 \sin F = \sin c \sin A_1,$$
$$\sin a_2 \sin F = \sin b \sin A_2,$$
$$\sin a \sin f \sin F = \sin b \sin c \sin A,$$

and so, on substitution of these values, we get

$$\cot f = \cot b \frac{\sin A_1}{\sin A} + \cot c \frac{\sin A_2}{\sin A} \quad \text{.................(12)}$$

144. To illustrate the use of these formulae, we shall apply them to some special cases.

Length of the median and of the bisectors of an angle.

(1) Let M be the mid point of the arc BC, and let m denote the length of AM, which may be called a *median* of the triangle.

Making F coincide with M, we get

$$a_1 = \tfrac{1}{2}a, \qquad a_2 = \tfrac{1}{2}a,$$

and so equation (11) gives us

$$\cos m = \frac{1}{2\cos\frac{1}{2}a}(\cos b + \cos c). \quad \ldots\ldots\ldots\ldots(13)^*$$

(2) Let AF be the internal bisector of the angle A, and let its length be denoted by η. Then $A_1 = A_2 = \frac{1}{2}A$, and from equation (12) we get

$$\cot\eta = \frac{1}{2\cos\frac{1}{2}A}(\cot b + \cot c). \quad \ldots\ldots\ldots\ldots(14)$$

(3) Let AF be the external bisector of the angle A, F being in BC produced, and let its length be denoted by η'. Then $A_1 = \frac{1}{2}\pi + \frac{1}{2}A$, $A_2 = -(\frac{1}{2}\pi - \frac{1}{2}A)$, and formula (12) yields

$$\cot\eta' = \frac{1}{2\sin\frac{1}{2}A}(\cot b - \cot c). \quad \ldots\ldots\ldots\ldots(15)$$

145. Relation between the arcs joining three points on a great circle and any other point.

The result (11) of Art. 143 may be regarded as a relation connecting the angular distances of three points B, F, and C, on the same great circle, and any other point A on the sphere. It may, indeed, be used as a test to determine whether three given points do or do not lie on the same great circle.

Looked at from this point of view the formula is equivalent to the following theorem, in the enunciation of which we introduce a fresh notation, in order to present the result in a symmetrical form.

---

* GUDERMANN, *Niedere Sphärik*, § 400.

If X, Y, Z *be three points on a great circle, and* P *any other point,*

$$\cos PX \sin YZ + \cos PY \sin ZX + \cos PZ \sin XY = 0, \quad \ldots(11a)$$

it being understood that arcs measured on the great circle in one direction are considered positive, those in the other direction negative, so that $ZY = -YZ$.

Instead of the cosines of PX, PY, PZ we may use the sines of the arcs drawn through X, Y, Z perpendicular to the great circle of which P is a pole, and thus we get a relation connecting the spherical perpendiculars drawn from three points on the same great circle to any other great circle. We shall re-state the resulting theorem in a form which exhibits its analogy to a well-known theorem in plane geometry; and append a proof which does not depend on Art. 143.

146. Relation between the arcs drawn perpendicular to a great circle from three points on another great circle.

From two points P_1 and P_2 arcs are drawn perpendicular to a fixed arc; and from a point P on the same great circle as P_1 and P_2 a perpendicular is drawn to the same fixed arc. Let $P_1P = \theta_1$ and $PP_2 = \theta_2$; and let the perpendiculars drawn from P, P_1, and P_2 be denoted by x, x_1, and x_2. Then will

$$\sin x = \frac{\sin \theta_2}{\sin(\theta_1 + \theta_2)} \sin x_1 + \frac{\sin \theta_1}{\sin(\theta_1 + \theta_2)} \sin x_2.$$

Let the arc P_2P_1, produced if necessary, cut the fixed arc at a point O; let α denote the angle between the arcs. We shall suppose that P_1 is between O and P_2, and that P is between P_1 and P_2.

Then, by Art. 73,

$$\sin x_1 = \sin \alpha \sin OP_1 = \sin \alpha \sin(OP - \theta_1)$$
$$= \sin \alpha (\sin OP \cos \theta_1 - \cos OP \sin \theta_1);$$
$$\sin x_2 = \sin \alpha \sin OP_2 = \sin \alpha \sin(OP + \theta_2)$$
$$= \sin \alpha (\sin OP \cos \theta_2 + \cos OP \sin \theta_2).$$

Multiply the former by $\sin\theta_2$, and the latter by $\sin\theta_1$, and add; thus

$$\sin\theta_2\sin x_1+\sin\theta_1\sin x_2=\sin(\theta_1+\theta_2)\sin a\sin \mathrm{OP}$$
$$=\sin(\theta_1+\theta_2)\sin x. \quad\ldots\ldots\ldots\ldots\ldots (11b)$$

The student should convince himself by examination that the result holds for all relative positions of P, P_1, and P_2, when due regard is paid to algebraical signs.

Examples of the application of this result will be found in the chapter on HART'S Circle.

The theorem of Plane Geometry analogous to that of the present article is as follows.

If P_1, P, P_2 be three points in the same straight line, and x_1, x, x_2 the lengths of the perpendiculars drawn from them to any other straight line, then

$$x=\frac{\mathrm{PP}_2}{\mathrm{P}_1\mathrm{P}_2}x_1+\frac{\mathrm{P}_1\mathrm{P}}{\mathrm{P}_1\mathrm{P}_2}x_2.$$

147. *The arc joining the mid points of the sides of a triangle intersects the base produced in points which are equidistant from the mid point of the base.*

Let L, M, N be the mid points of the sides, and let the arc LM intersect the base in the points X, Y, which are of course diametrically opposite.

Let AP, BQ, and CR be arcs drawn perpendicular to the arc ML from A, B, and C. These will meet in a point O, the pole of the circle LM.

In the triangles APM, CRM we have

$$\mathrm{A\hat{M}P}=\mathrm{C\hat{M}R}, \text{ vertically opposite angles,}$$

$$\mathrm{A\hat{P}M}=\mathrm{C\hat{R}M}, \text{ both being right angles,}$$

$$\mathrm{AM}=\mathrm{MC}, \text{ by hypothesis.}$$

Hence the triangles are equal in all respects.

Similarly the triangles BQL, CRL are equal; therefore

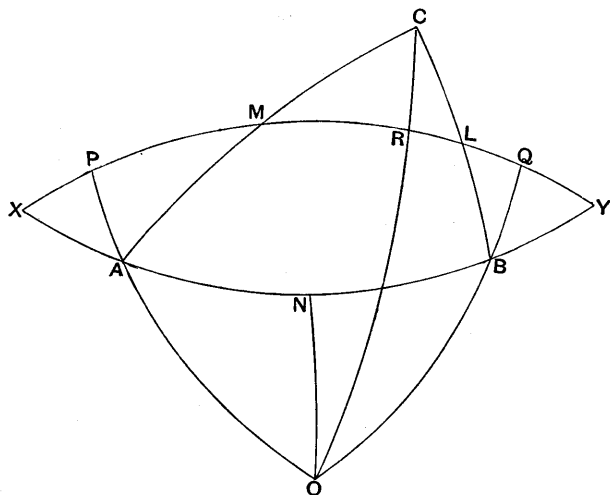
$$\mathrm{AP}=\mathrm{CR}=\mathrm{BQ}.$$

In the triangles APX, BQY, we now have

$$AP = BQ, \quad \hat{X} = \hat{Y}, \quad A\hat{P}X = B\hat{Q}Y;$$

therefore the triangles are symmetrically equal.

Accordingly AX = BY; and as AN = BN, we get finally

$$XN = YN = \tfrac{1}{2}X\hat{N}Y = \text{a quadrant}.$$

Corollary. If the base of a triangle be given, and the vertex variable, the arc joining the mid points of the sides passes constantly through two fixed points.

148. *The pole of the great circle joining the mid points of two sides is also the pole of the circumcircle of the colunar triangle.**

Using the figure of the previous Article, and denoting by p the length of the equal arcs AP, BQ, CR, we notice that

$$OC = OR + RC = \tfrac{1}{2}\pi + p, \quad OB = OA = OP - AP = \tfrac{1}{2}\pi - p;$$

* BALTZER, *Trigonometrie*, § 5.

hence $$OA = OB = \pi - OC = OC',$$
where C' is the point diametrically opposite to C.

Thus O is equidistant from the corners of the colunar triangle, and is therefore the pole of its circumcircle.

It is now easy to find the value of the radius R_3 of this circle. For $R_3 = \frac{1}{2}\pi - p$; and in the triangle CML

$$\sin p \sin ML = \sin CM \sin CL \sin C. \quad \ldots\ldots\ldots\ldots(16)$$

Also $$ML = \tfrac{1}{2} PQ = \tfrac{1}{2} A\hat{O}B = A\hat{O}N;$$

therefore $$\sin AON \cos R_3 = \sin \tfrac{1}{2}a \sin \tfrac{1}{2}b \sin C.$$

And in the right-angled triangle ANO

$$\sin A\hat{O}N \sin R_3 = \sin \tfrac{1}{2}c;$$

hence

$$\tan R_3 = \frac{\sin \frac{1}{2}c}{\sin \frac{1}{2}a \sin \frac{1}{2}b \sin C} = \frac{2}{n} \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \sin \tfrac{1}{2}c. \quad \ldots(17)$$

Substituting in this formula the elements of the colunar triangle, we deduce the value of the circum-radius of the original triangle, namely

$$\tan R = \frac{2}{n} \sin \tfrac{1}{2}a \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c. \quad \ldots\ldots\ldots\ldots\ldots(18)$$

These results agree with those obtained in Chapter VI.

149. Geometrical representation of the spherical excess. Still making use of the same figure, and remembering the pairs of triangles proved equal in Art. 147, we see that

$$X\hat{A}P = Y\hat{B}Q, \qquad M\hat{A}P = M\hat{C}R, \qquad L\hat{B}Q = L\hat{C}R; \quad \ldots\ldots(19)$$

hence

$$\begin{aligned} 2\,X\hat{A}P &= X\hat{A}C - M\hat{A}P + Y\hat{B}C - L\hat{B}Q, \\ &= \pi - A + \pi - B - C, \\ &= \pi - E, \text{ where } E \text{ is the spherical excess.} \quad \ldots\ldots(20) \end{aligned}$$

Thus $X\hat{A}P$ or $Y\hat{B}Q$ is the complement of half the spherical excess.

Now in the right-angled triangle APX,

$$\cos X\hat{A}P = \tan AP \cot AX;$$

therefore $$\sin \tfrac{1}{2}E = \tan \tfrac{1}{2}c \cot R_3. \quad \text{......(21)}$$

Substitute for cot R_3 from formula (17), and there results

$$\sin \tfrac{1}{2}E = \frac{n}{2 \cos \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c}, \quad \text{......(22)}$$

which is CAGNOLI'S formula.

From this and (18) we also get

$$\sin \tfrac{1}{2}E \cot R = \frac{2n^2}{\sin a \sin b \sin c} = N. \quad \text{......(23)}$$

150. Again in the triangles OAN and CML

$$\cos AN \sin O\hat{A}N = \cos A\hat{O}N = \cos ML,$$

and $$\cos ML = \cos CL \cos CM + \sin CL \sin CM \cos C;$$

hence $$\cos \tfrac{1}{2}E = \sin O\hat{A}N$$

$$= \frac{\cos \frac{1}{2}a \cos \frac{1}{2}b + \sin \frac{1}{2}a \sin \frac{1}{2}b \cos C}{\cos \frac{1}{2}c}, \quad \text{......(24)}$$

as in Art. 138, (26).

151. *If one angle of a triangle be given, and the product of the tangents of the halves of the sides containing the angle be constant, the area of the triangle is constant.*

If one side of a triangle be given, and the product of the tangents of the halves of the adjacent angles be constant, the perimeter of the triangle is constant.

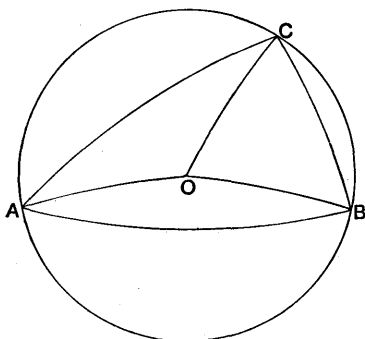
The first of these theorems is an immediate inference from the expression for cot $\frac{1}{2}E$ contained in formula (36), Art. 140. The second is obtained by applying the first to the polar triangle.

152. *If the base of a triangle be given, and the difference between the vertical angle and the sum of the base angles, then the circum-circle is known.*

Let O be the pole of the circumcircle. Join OA, OB, OC. Since the angles at the base of an isosceles triangle are equal,

$$O\hat{C}A = O\hat{A}C, \qquad O\hat{C}B = O\hat{B}C, \qquad O\hat{A}B = O\hat{B}A.$$

Therefore $\quad A + B - C = O\hat{A}B + O\hat{B}A = 2\,O\hat{A}B = 2\,O\hat{B}A \quad \text{.........(25)}$

So when $A + B - C$ is given, the angles $O\hat{A}B$ and $O\hat{B}A$ are given; and accordingly O is a known point and the circum-circle is completely known.

The result may be stated thus. When A and B are fixed, and $A + B - C$ constant, the locus of C is a small circle through A and B.

153. Lexell's locus.* *The base and the area of a spherical triangle being given, the locus of the vertex is a small circle.*

Let AB be the given base, and let A′, B′ be the points diametrically opposite to A and B. Then the triangle A′B′C has its angles A′ and B′ equal to $\pi - A$ and $\pi - B$ respectively. Thus

$$A' + B' - C = 2\pi - A - B - C$$
$$= \pi - E = \text{const.}$$

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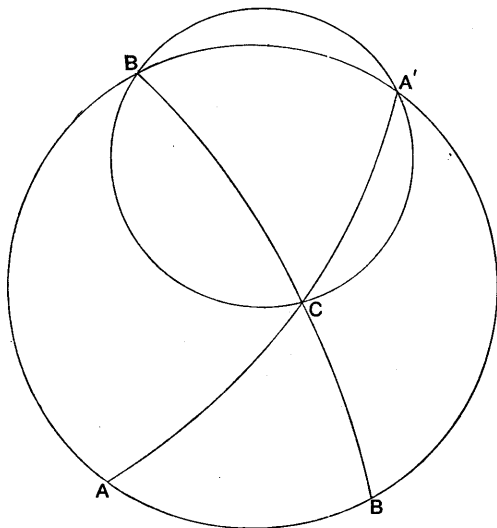
* LEXELL, *Acta Petropolitana*, 1781, I, p. 112.

Therefore, by the previous proposition, the locus of C is a small circle through A′ and B′.*

154. Keogh's theorem.† *The sine of half the spherical excess is equal to twice the norm of the sides of the triangle whose corners are the mid points of the original triangle.*

Referring to the figure of Art. 147, we have in the triangle AXP,

$$\cos X\hat{A}P = \sin X \cos XP,$$

that is, $$\sin \tfrac{1}{2}E = \sin X \sin ML. \quad\ldots\ldots\ldots\ldots\ldots\ldots(26)$$

But X is equal to the arc joining the mid points of the two

* By the use of LEXELL's locus it is possible to describe a spherical polygon of $n-1$ sides having the same area (or the same perimeter) as a given spherical polygon of n sides. See STEINER, *Crelle's Journal*, II, p. 45, and GUDERMANN, *Niedere Sphärik*, § 104.

† *Nouvelles Annales de Mathématiques*, 1st series, XVI, 1857, p. 142.

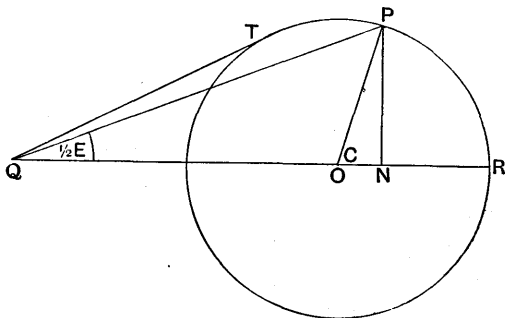
great semicircles XMY, XNY; that is, the arc drawn from N perpendicular to LM. And the product of the sine of this perpendicular and the sine of ML is twice n', the norm of the sides of the triangle LMN.

Hence $$\sin \tfrac{1}{2}E = 2n'. \qquad\qquad (27)$$

155. *To find the triangle of maximum area, two of whose sides are given.**

Let a, b be the given sides. Then, by Art. 138, formula (29),

$$\cot \tfrac{1}{2}E = \frac{\cot \frac{1}{2}a \cot \frac{1}{2}b + \cos C}{\sin C}. \qquad\qquad (28)$$

The greater the area the greater is E, and the less $\cot \frac{1}{2}E$; we have therefore to find what value of C, if any, makes the expression on the right side of this equation a minimum. This is done by a geometrical construction, as follows.

In any plane, with centre O and radius unity, describe a circle. In the same plane take a point Q whose rectilinear distance QO from O is equal to the known quantity $\cot \frac{1}{2}a \cot \frac{1}{2}b$. Produce QO to R, and make the angle ROP equal to the angle C of the spherical triangle, P being on the circumference of the unit circle. Draw PN perpendicular to QR.

* Legendre, *Géométrie*, VII, Prop. 26.

Then

$$\mathsf{NP} = \sin \mathsf{C}, \quad \mathsf{QN} = \mathsf{QO} + \mathsf{ON} = \cot \tfrac{1}{2}a \cot \tfrac{1}{2}b + \cos \mathsf{C},$$

and therefore, by equation (28) above,

$$\widehat{\mathsf{OQP}} = \tfrac{1}{2}\mathsf{E}. \quad \text{.......................} (29)$$

If Q lie outside the circle as in the diagram, that is, if $\cot \frac{1}{2}a \cot \frac{1}{2}b > 1$, or $a + b < \pi$, the greatest possible value of $\widehat{\mathsf{OQP}}$ occurs when QP is a tangent to the circle. If its position in that case be QT, we see that

$$\widehat{\mathsf{ROT}} = \tfrac{1}{2}\pi + \widehat{\mathsf{OQT}},$$

i.e.,

$$\mathsf{C} = \tfrac{1}{2}\pi + \tfrac{1}{2}\mathsf{E},$$

or

$$\mathsf{A} + \mathsf{B} = \mathsf{C}. \quad \text{.......................} (30)$$

The triangle of greatest area is accordingly that in which the angle contained by the given sides is equal to the sum of the remaining angles.

If $a + b = \pi$, $\cot \frac{1}{2}a \cot \frac{1}{2}b = 1$, and Q lies on the circumference of the unit circle. And as the angle ROP at the centre is double the angle RQP at the circumference,

$$\mathsf{C} = \mathsf{E},$$

or

$$\mathsf{A} + \mathsf{B} = \pi. \quad \text{.......................} (31)$$

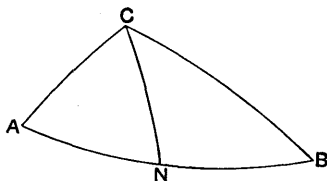
The greatest value of E is the same as the greatest value of C, namely π; and the limiting form of the triangle as its area increases is a lune whose angle is a right angle.

If $a + b > \pi$, $\cot \frac{1}{2}a \cot \frac{1}{2}b < 1$, and Q lies inside the unit circle. The angle OQP may now have any value from zero to π, and therefore E may have any value up to 2π. There is no real maximum; the triangle may be increased in area until it becomes a hemisphere.

Thus if $a + b \geq \pi$, there is no true maximum, but there is in each case a superior limit to the area. The area never quite attains the limiting value, as such attainment would involve the triangle's ceasing to be a triangle in the strict sense of the term.

156. *If in a spherical triangle the angle* C *be equal to the sum of the other two angles, the pole of the circumcircle will lie in the side* AB.

Draw the arc CN making the angle ACN equal to A, and let CN meet AB in N.

Then because $C = A + B$, and $N\hat{C}A = A$, the remainder $N\hat{C}B = B$.

Hence the triangles ANC, BNC are isosceles, so that

$$NA = NC = NB;$$

therefore N is the mid point of AB, and is the pole of the circum-circle.

157. Theorems concerning polygons. From the last two articles the following theorems may be deduced.

When all the sides but one of a spherical polygon are given in length, its area is greatest when the corners lie on a small circle whose pole is the mid point of the remaining side.

When all the sides of a spherical polygon are given in length, its area is greatest when it can be inscribed in a small circle.

In both these theorems it is necessary that the sides should be so small as always to justify the application of the first part of the theorem of Art. 155.

The results are proved in the same manner as the analogous propositions for plane polygons. See LEGENDRE'S *Géométrie*, VII, Prop. 27, or TOWNSEND'S *Modern Geometry*, Vol. I, p. 63.

158. *The arcs which bisect internally the angles of a spherical triangle are concurrent.**

* First proved by MENELAUS.

The point of concurrence is the pole of the inscribed circle. The proof is contained in Art. 119.

A corresponding result holds in the case of the internal bisector of one angle and the external bisectors of the other two.

The arcs drawn perpendicular to the sides of a triangle at their mid points are concurrent.

The point of concurrence is the pole of the circumcircle. The proof is contained in Art. 122.

159. *If three arcs pass through one point, the ratio of the sines of the arcs drawn from a variable point on one, perpendicular to the other two, is constant.*

Let the three arcs be OA, OB, OC. Take any point P in OB, and draw PM and PN perpendicular to OA and OC respectively.

Then in the right-angled triangles PMO, PNO,

$$\sin PM = \sin OP \sin A\hat{O}B, \quad \sin PN = \sin OP \sin C\hat{O}B;$$

therefore
$$\frac{\sin PM}{\sin PN} = \frac{\sin AOB}{\sin COB}.$$

Thus the ratio of sin PM to sin PN is independent of the position of P on the arc OB.

Conversely, suppose that from any other point p arcs pm and pn are drawn perpendicular to OA and OC respectively; then if

$$\frac{\sin pm}{\sin pn} = \frac{\sin PM}{\sin PN},$$

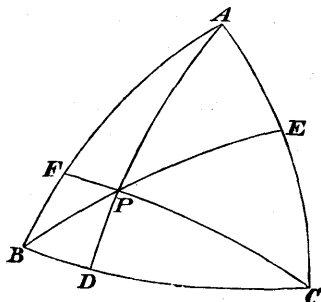
it will follow that p is on the same great circle as O and P.

160. *If three arcs be drawn from the angles of a spherical triangle through any point to meet the opposite sides, the products of the sines of the alternate segments of the sides are equal.*

Let P be any point, and let arcs be drawn from the angles

A, B, C passing through P and meeting the opposite sides at D, E, F. Then

$$\frac{\sin BD}{\sin BP}=\frac{\sin B\hat{P}D}{\sin B\hat{D}P}, \qquad \frac{\sin DC}{\sin CP}=\frac{\sin C\hat{P}D}{\sin C\hat{D}P},$$

therefore
$$\frac{\sin BD}{\sin DC}=\frac{\sin B\hat{P}D}{\sin C\hat{P}D}\,\frac{\sin BP}{\sin CP}.$$

Similar expressions may be found for $\frac{\sin CE}{\sin EA}$ and $\frac{\sin AF}{\sin FB}$;

also $\quad B\hat{P}D = A\hat{P}E, \quad D\hat{P}C = A\hat{P}F, \quad C\hat{P}E = B\hat{P}F,$

and hence it follows obviously that

$$\frac{\sin BD}{\sin DC}\,\frac{\sin CE}{\sin EA}\,\frac{\sin AF}{\sin FB}=1\,; \quad \ldots\ldots\ldots\ldots\ldots\ldots(32)$$

therefore $\sin BD \sin CE \sin AF = \sin DC \sin EA \sin FB.$ ......(33)

This theorem is analogous to CEVA'S theorem * for a plane triangle.

Conversely, when the points D, E, F in the sides of a spherical triangle are such that the relation given in formula (33) holds, the arcs which join these points with the opposite corners respectively *pass through a common point.* Hence the following propositions may be established :

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* See NIXON'S *Euclid Revised*, General Addenda, Section II, or RUSSELL'S *Pure Geometry*, Chapter I.

The perpendiculars from the corners of a spherical triangle on the opposite sides meet at a point;*

The arcs which bisect the angles of a spherical triangle meet at a point;

The arcs which join the corners of a spherical triangle with the middle points of the opposite sides meet at a point;

The arcs which join the corners of a spherical triangle with the points where the inscribed circle touches the opposite sides respectively meet at a point.

161. Spherical mean centre. In connection with the theorem of the preceding Article it is of interest to notice that, if $2n_1$, $2n_2$, $2n_3$ be the sines† of the triangles BPC, CPA, APB respectively,

$$\sin \mathrm{BD} : \sin \mathrm{DC} = n_3 : n_2. \quad \text{..................(34)}$$

For, if β and γ be the perpendiculars drawn from B and C to AD,

$$2n_3 = \sin\beta \sin \mathrm{AP}, \quad 2n_2 = \sin\gamma \sin \mathrm{AP}\,; \quad \text{..........(35)}$$

but $\quad \sin\beta = \sin \mathrm{BD} \sin \mathrm{D}, \quad$ and $\quad \sin\gamma = \sin \mathrm{DC} \sin \mathrm{D}.$

Hence $\quad \sin \mathrm{BD} : \sin \mathrm{DC} = \sin\beta : \sin\gamma = n_3 : n_2.$

On account of the analogy which exists between this property and a property of the mean centre of three points on a plane, Dr. CASEY calls P the *spherical mean centre* of the points A, B, C for multiples n_1, n_2, n_3 respectively.

It is readily seen that n_1, n_2, n_3 are analogous to the areas BPC, CPA, APB in Plane Geometry, or to the *areal coordinates* of P. In fact n_1/n, n_2/n, n_3/n are *volumetric coordinates* of P with respect to the tetrahedron whose corners are A, B, C, and the centre of the sphere. See second foot-note on page 29.

162. Normal coordinates of a point with respect to a triangle. If x, y, z denote the lengths of the arcs drawn

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* GUDERMANN, *Niedere Sphärik*, § 68; SCHULZ, *Sphärik*, II, § 47.

† See Art. 51.

from a point P perpendicular to the sides of the triangle ABC, then $\sin x$, $\sin y$, $\sin z$ are called *the normal coordinates of* P *with respect to the triangle.* When the ratios of the coordinates are given, the point is determined.

Normal coordinates are clearly analogous to *trilinear coordinates* with respect to a plane triangle.

Suppose, for example, that P is any point on the arc AD drawn perpendicular to BC. See figure of Art. 160.

Then in the right-angled triangles ADB, ADC, we have, by Art. 73,

$$\cos B = \cos AD \sin D\hat{A}B, \quad \cos C = \cos AD \sin D\hat{A}C;$$

therefore

$$\frac{\cos C \cos A}{\cos A \cos B} = \frac{\cos C}{\cos B} = \frac{\sin DAC}{\sin DAB} = \frac{\sin y}{\sin z}; \quad \text{(Art. 159).}$$

If P also lie on the arc BE drawn perpendicular to CA, we have likewise

$$\frac{\cos A \cos B}{\cos B \cos C} = \frac{\sin z}{\sin x};$$

hence it follows that

$$\frac{\cos B \cos C}{\cos C \cos A} = \frac{\sin x}{\sin y},$$

and this shews that P is on the arc drawn from C perpendicular to AB.

Thus the three perpendiculars meet at a point, and this point is determined by the relations

$$\frac{\sin x}{\cos B \cos C} = \frac{\sin y}{\cos C \cos A} = \frac{\sin z}{\cos A \cos B}. \quad \text{............(36)}$$

In the same manner it may be shewn that the arcs drawn from the angles of a spherical triangle to the middle points of the opposite sides meet at a point; and if from this point arcs x, y, z are drawn perpendicular to the sides a, b, c respectively,

$$\frac{\sin x}{\sin B \sin C} = \frac{\sin y}{\sin C \sin A} = \frac{\sin z}{\sin A \sin B}. \quad \text{............(37)}$$

163. It should be noticed that the normal coordinates of any point are connected with the sines of the triangles PBC, PCA, PAB by the relations

$$\sin x \sin a = 2n_1, \quad \sin y \sin b = 2n_2, \quad \sin z \sin c = 2n_3. \quad \ldots\ldots\ldots (38)$$

The three coordinates of a point are not independent of one another; they must always be such as to satisfy the relation

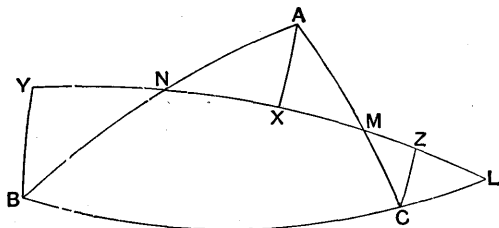
$$n_1^2 + n_2^2 + n_3^2 + 2n_2n_3\cos a + 2n_3n_1\cos b + 2n_1n_2\cos c = n^2. \quad \ldots\ldots\ldots (39)$$

An easy proof of this is obtained by writing down the volumetric equation of the sphere (cf. Art. 161).

164. Property of a spherical transversal. *If a great circle intersect the sides of a triangle* ABC *in* L, M, *and* N,

$$\frac{\sin BL}{\sin LC}\,\frac{\sin CM}{\sin MA}\,\frac{\sin AN}{\sin NB} = -1.$$

Either *one* or *all three* of the sides of the triangle will be cut externally by the arc LMN. The figure represents the case when only BC is cut externally.

Draw the arcs AX, BY, CZ perpendicular to LMN.

From the right-angled triangles BYL, CZL,

$$\sin BY = \sin BL \sin Y\hat{L}B, \quad \sin CZ = \sin CL \sin Y\hat{L}B;$$

hence

$$\frac{\sin BL}{\sin CL} = \frac{\sin BY}{\sin CZ}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (40)$$

Similarly from the pairs of triangles CMZ, AMX, and ANX, BNY,

$$\frac{\sin CM}{\sin MA} = \frac{\sin CZ}{\sin AX}, \quad \frac{\sin AN}{\sin NB} = \frac{\sin AX}{\sin BY}. \quad \ldots\ldots\ldots\ldots (41)$$

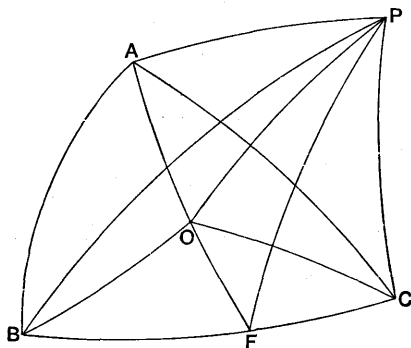
Multiplying corresponding sides of these three equations, and writing $-\sin \mathsf{LC}$ for $\sin \mathsf{CL}$, we get

$$\frac{\sin \mathsf{BL}}{\sin \mathsf{LC}}\,\frac{\sin \mathsf{CM}}{\sin \mathsf{MA}}\,\frac{\sin \mathsf{AN}}{\sin \mathsf{NB}} = -1. \quad\text{..............} \quad (42)$$

This theorem is analogous to MENELAUS' Theorem * for a plane triangle cut by a transversal.

The converse also is true, namely that, if L, M, N be points on the sides of a triangle for which relation (42) holds good, they lie on a great circle.

165. *If* O *be any point within a spherical triangle* ABC, *and* P *any other point on the sphere; and if* $2n_1$, $2n_2$, $2n_3$ *be the sines of the triangles* OBC, OCA, OAB; *then*

$$n_1 \cos \mathsf{PA} + n_2 \cos \mathsf{PB} + n_3 \cos \mathsf{PC} = n \cos \mathsf{PO}.\dagger$$

Produce AO to meet BC in F.

Apply to the points P, B, F, C the theorem of Art. 143 (11), and we get

$$\sin \mathsf{FC} \cos \mathsf{PB} + \sin \mathsf{BF} \cos \mathsf{PC} = \sin \mathsf{BC} \cos \mathsf{PF}.$$

* See NIXON or RUSSELL, *loc. cit.*

† This theorem is due to Dr. CASEY (*Spherical Trigonometry*, p. 81).

Applying the same theorem to the points P, A, O, F we get

$$\sin \mathrm{FO} \cos \mathrm{PA} + \sin \mathrm{OA} \cos \mathrm{PF} = \sin \mathrm{FA} \cos \mathrm{PO}.$$

The elimination of cos PF from these results gives

$$\sin \mathrm{FO} \sin \mathrm{BC} \cos \mathrm{PA} + \sin \mathrm{OA} \sin \mathrm{FC} \cos \mathrm{PB} + \sin \mathrm{OA} \sin \mathrm{BF} \cos \mathrm{PC} = \sin \mathrm{BC} \sin \mathrm{FA} \cos \mathrm{PO}.$$

If we multiply this through by sin F, and notice that

$$\sin \mathrm{BC} \sin \mathrm{FO} \sin \mathrm{F} = 2n_1, \quad \sin \mathrm{OA} \sin \mathrm{FC} \sin \mathrm{F} = 2n_2,$$
$$\sin \mathrm{OA} \sin \mathrm{BF} \sin \mathrm{F} = 2n_3, \quad \sin \mathrm{BC} \sin \mathrm{FA} \sin \mathrm{F} = 2n,$$

we get finally

$$n_1 \cos \mathrm{PA} + n_2 \cos \mathrm{PB} + n_3 \cos \mathrm{PC} = n \cos \mathrm{PO}. \quad \text{.......(43)}$$

166. *If the base* BC *of a spherical triangle be given, and the ratio* $\sin \frac{1}{2}b : \sin \frac{1}{2}c$, *the locus of* A *is a small circle.*

Since the ratio $\sin \frac{1}{2}b : \sin \frac{1}{2}c$ is given, the ratio of the straight lines joining A to B and to C is given. Therefore A must lie on a certain sphere with respect to which B and C are inverse points. The locus of A is the curve of intersection of this sphere with the original sphere, namely a circle.

167. *If two spherical triangles have their vertical angles equal, and the difference of the base angles of the one equal to the difference of the base angles of the other, then the ratio of the tangents of the halves of the sides opposite the base angles of the one is equal to the corresponding ratio for the other.*

Let the elements of the second triangle be denoted by accented letters, and let C, C′ be the equal vertical angles.

From the second of Napier's analogies we have

$$\frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} = \tan \tfrac{1}{2}(\mathrm{A}-\mathrm{B}) \tan \tfrac{1}{2}\mathrm{C}$$
$$= \tan \tfrac{1}{2}(\mathrm{A}'-\mathrm{B}') \tan \tfrac{1}{2}\mathrm{C}'$$
$$= \frac{\sin \frac{1}{2}(a'-b')}{\sin \frac{1}{2}(a'+b')}.$$

Hence

$$\frac{\sin\frac{1}{2}(a+b)-\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)+\sin\frac{1}{2}(a-b)}=\frac{\sin\frac{1}{2}(a'+b')-\sin\frac{1}{2}(a'-b')}{\sin\frac{1}{2}(a'+b')+\sin\frac{1}{2}(a'-b')},$$

or

$$\frac{\tan\frac{1}{2}b}{\tan\frac{1}{2}a}=\frac{\tan\frac{1}{2}b'}{\tan\frac{1}{2}a'}. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(44)$$

Triangles so related have a certain analogy to *similar* plane triangles.

EXAMPLES XI.

1. Solve the triangle having given the sides a, b, and the median m_3.

2. Solve the triangle having given the sides a, b, and the median m_1.

3. Solve the triangle having given the angle A and the segments into which the side b is divided by the bisector of the opposite angle.

4. If the base AB of a spherical triangle ABC be produced to a point O such that

$$\tan \mathrm{BO}=\frac{\cos(a+b)\sin c}{1-\cos c\cos(a+b)},$$

shew that

$$\sin c\cos \mathrm{CO}=\sin(a+b)\sin b\sin \mathrm{AO}.$$

Hence shew that if the base of a spherical triangle be fixed, and the vertex move so that the sum of the sides remains constant, a point may be found on the base produced, such that the cosine of the distance of the vertex from this point bears a constant ratio to the cosine of the distance of the vertex from a certain fixed great circular arc.

(R. U. I., 1899.)

5. P, Q, R are the feet of the perpendiculars from the vertices A, B, C of a spherical triangle on the opposite sides: prove that BQ is a bisector of the angle PQR; and shew that, when the triangle ABC is acute-angled,

$$\cos \mathrm{P\hat{Q}R}=\frac{\cos^2\mathrm{A}-\cos^2\mathrm{B}+\cos^2\mathrm{C}+2\cos\mathrm{A}\cos\mathrm{B}\cos\mathrm{C}}{\cos^2\mathrm{A}+\cos^2\mathrm{B}+\cos^2\mathrm{C}+2\cos\mathrm{A}\cos\mathrm{B}\cos\mathrm{C}}.$$

(R. U. I., 1898.)

6. If the triangle ABC be such that the small circle on AB as diameter passes through C, prove that $\cot \mathrm{A}\cot \mathrm{B}=\cos^2\frac{1}{2}c$. (R. U. I., 1898.)

7. If two sides of a spherical triangle be supplementary, prove that the median passing through their intersection is a quadrant.

(R. U. I., 1895.)

8. If the three internal bisectors of the angles A, B, C of a spherical triangle intersect in O, and meet the opposite sides in P, Q, and R respectively, prove that

$$\frac{\sin \mathrm{PO}}{\sin a \sin \mathrm{PA}} = \frac{\sin \mathrm{QO}}{\sin b \sin \mathrm{QB}} = \frac{\sin \mathrm{RO}}{\sin c \sin \mathrm{RC}}$$

$$= \frac{1}{\sqrt{\sin^2 s + \sin s \sin(s-a) \sin(s-b) \sin(s-c)}}.$$

(R. U. I., 1895.)

9. If the points where the internal and external bisectors of the vertical angle A of a spherical triangle meet the base be P and Q respectively, prove that

$$\cos \mathrm{BPA} = \tfrac{1}{2}(\cos \mathrm{C} - \cos \mathrm{B}) \sec \tfrac{1}{2}\mathrm{A},$$

and that

$$\cos \mathrm{PQ} = \frac{\sin(\mathrm{B}-\mathrm{C}) \sin(\mathrm{B}+\mathrm{C})}{\sqrt{(\cos^2\mathrm{B} + \cos^2\mathrm{C} - 2)^2 - 4(\cos \mathrm{A} + \cos \mathrm{B} \cos \mathrm{C})^2}}.$$

(R. U. I., 1895.)

10. ABC is a spherical triangle, E is the mid point of BC, and AD is drawn at right angles to BC; shew that

$$\tan \mathrm{ED} \sin(\mathrm{B}+\mathrm{C}) = \tan \tfrac{1}{2}a \sin(\mathrm{B}-\mathrm{C}).$$

Putting out of the question all cases in which ED exceeds a quadrant, find under what circumstances D and B are on the same side of E, and under what circumstances they are on opposite sides of E.

(Sci. and Art, 1894.)

11. ABCD is a spherical quadrangle, and E, F are the mid points of the arcs CA and DB. Shew that

$$\cos \mathrm{AB} + \cos \mathrm{BC} + \cos \mathrm{CD} + \cos \mathrm{DA} = 4 \cos \tfrac{1}{2} \mathrm{CA} \cos \tfrac{1}{2}\mathrm{DB} \cos \mathrm{EF}.$$

(GUDERMANN.)

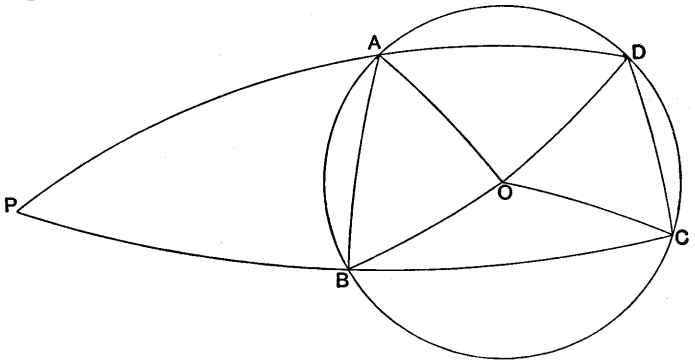
12. a, b, c, d are four great circles, and e, f are the great circles bisecting the angles (ca) and (db). Shew that

$$\cos(ab) + \cos(bc) + \cos(cd) + \cos(da) = 4 \cos \tfrac{1}{2}(ca) \cos \tfrac{1}{2}(db) \cos(ef).$$

(GUDERMANN.)

CHAPTER IX.

PROPERTIES OF CIRCLES ON THE SPHERE.

168. If the corners of a spherical quadrilateral lie on a small circle, the sums of its pairs of opposite angles are equal.*

Let ABCD be the quadrilateral, and O the pole of the small circle. Join O to each of the corners; then the triangles so formed are isosceles, and consequently

$$O\hat{D}A = O\hat{A}D, \quad O\hat{D}C = O\hat{C}D, \quad O\hat{B}A = O\hat{A}B, \quad O\hat{B}C = O\hat{C}B.$$

Adding corresponding sides of these equalities, we get

$$A\hat{D}C + A\hat{B}C = B\hat{A}D + B\hat{C}D. \quad \text{..................(1)}$$

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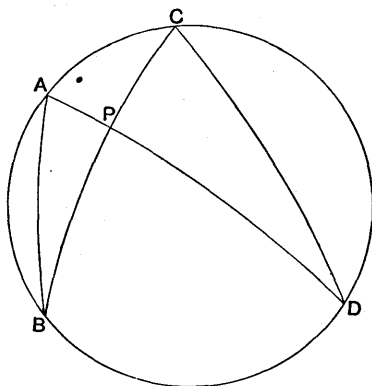
* LEXELL, *Acta Petropolitana*, 1782.

If the quadrilateral be crossed, as in the second figure, we find in a similar manner

$$\hat{BAD} + \hat{ADC} = \hat{ABC} + \hat{BCD}. \quad \text{.....................(2)}$$

Both cases are included in the following enunciation.

If A, B, C, D be four points on a small circle, and if the arcs AD, BC intersect in P, then

$$\hat{PCD} - \hat{PDC} = \hat{PAB} - \hat{PBA}. \quad \text{.....................(3)}$$

169. Spherical power of a point with respect to a circle. From the previous article it appears that the triangles PAB and PCD are related to one another in the same manner as the triangles discussed in Art. 167, whether P be inside or outside the small circle.

Hence
$$\frac{\tan \frac{1}{2} PB}{\tan \frac{1}{2} PA} = \frac{\tan \frac{1}{2} PD}{\tan \frac{1}{2} PC},$$

so that
$$\tan \tfrac{1}{2} PA \tan \tfrac{1}{2} PD = \tan \tfrac{1}{2} PB \tan \tfrac{1}{2} PC. \quad \text{............(4)}$$

Thus the product $\tan \frac{1}{2} PA \tan \frac{1}{2} PD$ is the same for all arcs such as PAD drawn through P.

To get a convenient expression for the constant value of this product, we have only to take as a particular position of

PAD the arc joining P to the pole O of the circle. Then if we denote the angular distance PO by δ, and the angular radius of the circle by ρ, the particular value of PD is $\rho+\delta$; and that of PA is $\delta-\rho$ when P is outside the circle, $\rho-\delta$ when P is inside.

Therefore, for P outside,

$$\tan\tfrac{1}{2}\text{PA}\tan\tfrac{1}{2}\text{PD}=\tan\tfrac{1}{2}(\delta+\rho)\tan\tfrac{1}{2}(\delta-\rho); \quad\text{.........(5)}$$

for P inside,

$$\tan\tfrac{1}{2}\text{PA}\tan\tfrac{1}{2}\text{PD}=\tan\tfrac{1}{2}(\rho+\delta)\tan\tfrac{1}{2}(\rho-\delta). \quad\text{.........(6)}$$

When A and D coincide, say at T, the arc PT touches the circle at T, and the length of the tangent arc is given by

$$\tan^2\tfrac{1}{2}\text{PT}=\tan\tfrac{1}{2}(\delta+\rho)\tan\tfrac{1}{2}(\delta-\rho). \quad\text{...........(7)}$$

The constant $\tan\frac{1}{2}(\delta+\rho)\tan\frac{1}{2}(\delta-\rho)$, or $\dfrac{\cos\rho-\cos\delta}{\cos\rho+\cos\delta}$, is called *the spherical power of the point* P *with respect to the circle.* It is positive when P is outside the circle, negative when P is inside.

The theorem of this Article is so important that we shall give another proof of it.

170. If a variable arc of a great circle, passing through a fixed point P, cut a given small circle in A and B,

$$\tan\tfrac{1}{2}\text{PA}\tan\tfrac{1}{2}\text{PB}$$

is constant.*

From O, the pole of the small circle, draw the arc OM perpendicular to AB.

Clearly the triangles OAM, OBM are symmetrically equal, and therefore M is the mid point of AB.

From the right-angled triangles POM, AOM

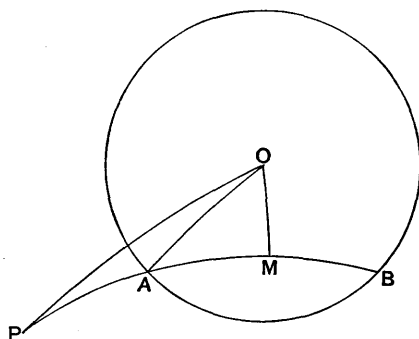
$$\frac{\cos\text{PM}}{\cos\text{PO}}=\frac{1}{\cos\text{MO}}=\frac{\cos\text{AM}}{\cos\text{AO}};$$

* Lexell, *Acta Petropolitana*, 1782, p. 65.

therefore

$$\frac{\cos \mathrm{AM} - \cos \mathrm{PM}}{\cos \mathrm{AM} + \cos \mathrm{PM}} = \frac{\cos \mathrm{AO} - \cos \mathrm{PO}}{\cos \mathrm{AO} + \cos \mathrm{PO}},$$

or,

$$\tan \tfrac{1}{2}(\mathrm{PM} - \mathrm{AM}) \tan \tfrac{1}{2}(\mathrm{PM} + \mathrm{AM}) = \tan \tfrac{1}{2}(\mathrm{PO} - \mathrm{AO}) \tan \tfrac{1}{2}(\mathrm{PO} + \mathrm{AO}).$$

This is the same as

$$\tan \tfrac{1}{2}\mathrm{PA} \tan \tfrac{1}{2}\mathrm{PB} = \tan \tfrac{1}{2}(\delta - \rho) \tan \tfrac{1}{2}(\delta + \rho), \quad \ldots\ldots (8)$$

the required result.

171. If from a variable point P on a fixed great circle PN the spherical tangents PE, PF be drawn to a given small circle, the product $\tan \frac{1}{2}\mathrm{NPE} \cot \frac{1}{2}\mathrm{NPF}$ has a constant value.*

From O, the pole of the small circle, draw the arc ON perpendicular to PN.

Clearly the arc OP bisects the angle EPF.

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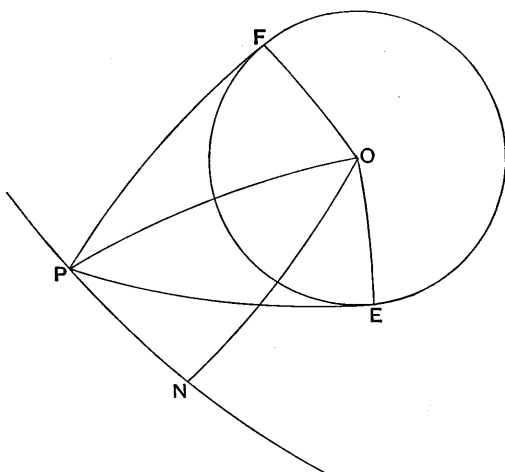
*GUDERMANN, *Niedere Sphärik*, § 296. This constant may be called the *spherical power of the great circle with respect to the small circle.* (See foot-note to Reviser's Preface.) If the great and small circles intersect at an angle ϕ, the spherical power is equal to $\tan^2 \frac{1}{2}\phi$.

In the right-angled triangles OEP, ONP

$$\frac{\sin \mathrm{EO}}{\sin \mathrm{EPO}} = \sin \mathrm{PO} = \frac{\sin \mathrm{NO}}{\sin \mathrm{NPO}};$$

therefore

$$\frac{\sin \mathrm{NPO} - \sin \mathrm{EPO}}{\sin \mathrm{NPO} + \sin \mathrm{EPO}} = \frac{\sin \mathrm{NO} - \sin \mathrm{EO}}{\sin \mathrm{NO} + \sin \mathrm{EO}},$$

or

$$\frac{\tan \frac{1}{2}(\mathrm{NPO} - \mathrm{EPO})}{\tan \frac{1}{2}(\mathrm{NPO} + \mathrm{EPO})} = \frac{\tan \frac{1}{2}(\mathrm{NO} - \mathrm{EO})}{\tan \frac{1}{2}(\mathrm{NO} + \mathrm{EO})}.$$

Denote NO by p, which is a constant. The equality is seen to be the same as

$$\frac{\tan \frac{1}{2}\mathrm{NPE}}{\tan \frac{1}{2}\mathrm{NPF}} = \frac{\tan \frac{1}{2}(p - \rho)}{\tan \frac{1}{2}(p + \rho)}. \quad \text{.....................(9)}$$

172. Relation between the arcs joining four points on a small circle.

Let the points, taken in order, be A, B, C, and D. By PTOLEMY'S theorem we have the following relation subsisting between the lengths of the chords,

$$\mathrm{AB}\,.\,\mathrm{CD} + \mathrm{AD}\,.\,\mathrm{BC} = \mathrm{AC}\,.\,\mathrm{BD}. \quad \text{...............(10)}$$

Now if R be the radius of the sphere

$$\text{chord AB} = 2\text{R} \sin \tfrac{1}{2}\text{AB}; \quad \ldots\ldots(11)$$

hence (10) may be written in the form

$$\sin \tfrac{1}{2}\text{AB} \sin \tfrac{1}{2}\text{CD} + \sin \tfrac{1}{2}\text{AD} \sin \tfrac{1}{2}\text{BC} = \sin \tfrac{1}{2}\text{AC} \sin \tfrac{1}{2}\text{BD}. \quad \ldots(12)$$

This is the required relation; it holds whether the circle be a small or a great circle. It must, of course, be understood that, whether the circle be great or small, AB, etc., represent arcs of *great* circles.

173. Relations between the mutual distances of four points on a great circle.

Let the points, taken in order, be A, B, C, and D.

If we put $\quad \text{AB} = \theta, \quad \text{BC} = \phi, \quad \text{CD} = \psi,$

then $\quad \text{AC} = \theta + \phi, \quad \text{BD} = \phi + \psi, \quad \text{AD} = \theta + \phi + \psi;$

and any identical relation that exists between the trigonometrical functions of θ, ϕ, ψ, $\theta + \phi$, $\phi + \psi$, $\theta + \phi + \psi$, holds equally for the corresponding functions of AB, BC, etc.

For example,

$$\sin \theta \sin \psi + \sin \phi \sin (\theta + \phi + \psi) \equiv \sin (\theta + \phi) \sin (\phi + \psi); \quad \ldots(13)$$

and therefore, for four points on a great circle,

$$\sin \text{AB} \sin \text{CD} + \sin \text{BC} \sin \text{AD} = \sin \text{AC} \sin \text{BD}. \quad \ldots\ldots(14)$$

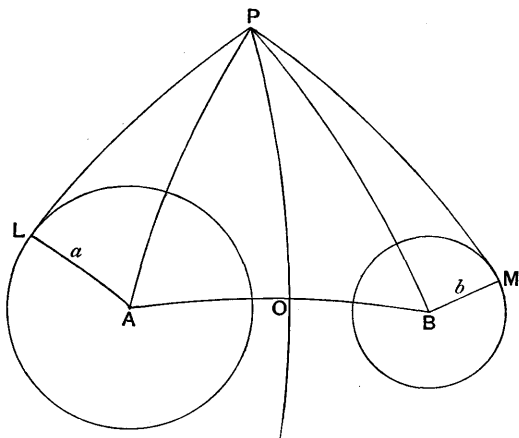
Similarly

$$\cos \text{AB} \cos \text{CD} - \cos \text{AC} \cos \text{BD} = \sin \text{BC} \sin \text{AD}. \quad \ldots\ldots(15)$$

An endless number of such relations may be found. Every relation between the trigonometrical functions of the arcs holds equally well for the same trigonometrical functions of any equimultiples or equi-submultiples of the arcs.

174. To find the locus of a point on a sphere such that the spherical tangents drawn from it to two given small circles are equal.

Let A, B be the poles of the small circles; a, b their angular radii; P a point such that the spherical tangents PL, PM drawn from it to the circles are equal.

Draw the great circle PO perpendicular to AB and meeting it in O.

Then, because the angles at L, M, and O are right angles,

$$\cos \mathrm{PL} = \frac{\cos \mathrm{PA}}{\cos a} = \frac{\cos \mathrm{AO}}{\cos a} \cos \mathrm{OP},$$

and
$$\cos \mathrm{PM} = \frac{\cos \mathrm{PB}}{\cos b} = \frac{\cos \mathrm{OB}}{\cos b} \cos \mathrm{OP}.$$

Hence, as PL = PM,

$$\frac{\cos \mathrm{AO}}{\cos a} = \frac{\cos \mathrm{OB}}{\cos b}, \quad \text{.....................(16)}$$

and therefore $\dfrac{\cos \mathrm{AO}}{\cos \mathrm{OB}} = \dfrac{\cos a}{\cos b} =$ a known constant. ....(17)

Thus O is a fixed point, and the locus of P is a great circle, namely that drawn through O at right angles to AB.

This great circle is called the *radical circle* of the two given circles. Clearly, if the given circles intersect, their radical circle passes through their common points; in this case part of the locus is within both circles, and the corresponding tangents

have no real existence. Remembering however that the square of the tangent of half the spherical tangent from a point to a circle is equal to the spherical power of the point with respect to the circle, we may define our locus as the locus of a point whose spherical powers with respect to two given circles are equal. Stated in this form, the property is intelligible for points inside as well as for points outside the circles.

175. Coaxal Circles.

Definition. *A system of circles on a sphere which is such that all pairs of circles of the system have the same radical circle is called a coaxal system of circles.*

If A, B, C, D, etc. be the poles, and a, b, c, d, etc. the radii ot the circles of a coaxal system, it is clear that:

(1) A, B, C, D, etc. must lie on the same great circle.

(2) There must be a certain point O (the *axial centre*) on this great circle, such that

$$\frac{\cos OA}{\cos a} = \frac{\cos OB}{\cos b} = \frac{\cos OC}{\cos c} = \frac{\cos OD}{\cos d} \text{ etc.} \quad \ldots\ldots\ldots(18)$$

(3) The tangents to all these circles from any point P on the radical circle are equal; and a circle, with P as pole and the common length of tangent as radius, cuts all the circles of the system orthogonally.

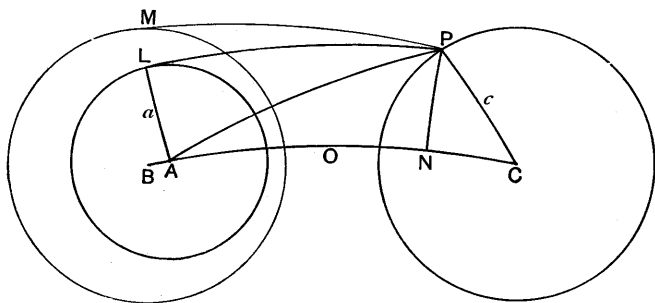
(4) If two of the circles intersect, then all circles of the system go through the same two points; this happens when the equal ratios of formula (18) are greater than unity.

(5) When the ratios of formula (18) are less than unity, equal to k say, there are two point-circles of the system. They are on opposite sides of O and equidistant from it, the the distance being $\cos^{-1}k$. These are called the *limiting points.*

(6) When two circles touch, the axial centre and the limiting points of the coaxal system to which they belong coincide at the point of contact, and all circles of the system touch one another at the same point. This happens when the

ratios of formula (18) are equal to unity; the corresponding case is one of transition between cases (4) and (5).

176. To find an expression for the length of the spherical tangent drawn to a small circle from a point on another small circle.

Let A and C be the poles of the small circles, a and c their radii, O their axial centre, P a point on the circle C, and PL a tangent arc to the circle A. Draw PN perpendicular to AC.

Using the cosine formula in the right-angled triangles PLA, PNA, PNC, we readily get

$$\cos a - \cos a \cos \mathrm{PL} = \cos a - \cos \mathrm{PA}$$

$$= \cos a - \frac{\cos \mathrm{AN}}{\cos \mathrm{NC}} \cos c$$

$$= \cos c \left\{ \frac{\cos \mathrm{AO}}{\cos \mathrm{OC}} - \frac{\cos \mathrm{AN}}{\cos \mathrm{NC}} \right\}$$

$$= \cos c \, . \, \frac{\sin \mathrm{AC} \sin \mathrm{ON}}{\cos \mathrm{OC} \cos \mathrm{NC}} \text{ (Art. 173, form. 15).}$$

Hence $$2 \sin^2 \tfrac{1}{2}\mathrm{PL} = \frac{\sin \mathrm{ON} \cos c}{\cos \mathrm{OC} \cos \mathrm{NC}} \cdot \frac{\sin \mathrm{AC}}{\cos a} \quad \text{.................. (19)}$$

177. If A, B, C be three circles of a coaxal system, and if

from a variable point P on C tangents PL, PM be drawn to A and B, then $\sin\frac{1}{2}$PL is to $\sin\frac{1}{2}$PM in a constant ratio.

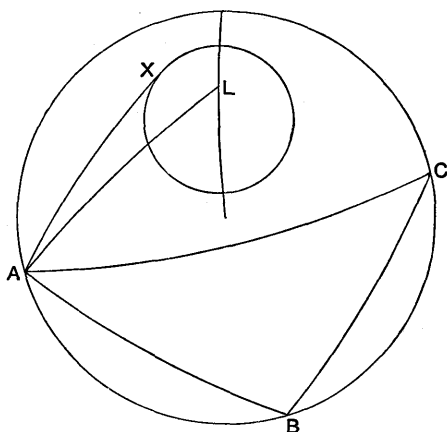
By the previous Article,

$$2\sin^2\tfrac{1}{2}\mathrm{PM} = \frac{\sin \mathrm{ON} \cos c}{\cos \mathrm{OC} \cos \mathrm{NC}} \cdot \frac{\sin \mathrm{BC}}{\cos b}; \qquad (20)$$

dividing sides of (19) by corresponding sides of (20), we get

$$\frac{\sin^2\frac{1}{2}\mathrm{PL}}{\sin^2\frac{1}{2}\mathrm{PM}} = \frac{\sin \mathrm{AC}}{\sin \mathrm{BC}} \cdot \frac{\cos b}{\cos a}. \qquad (21)$$

The right side of this equality is the same for all positions of P on the circle C, and therefore the left side is constant.

178. Condition for the contact of a given circle with the circumcircle of a given triangle.

A, B, C are three points on a circle; AX, BY, CZ are spherical tangents to another circle; then if, of the three products

$$\sin\tfrac{1}{2}\mathrm{AX}\sin\tfrac{1}{2}\mathrm{BC}, \quad \sin\tfrac{1}{2}\mathrm{BY}\sin\tfrac{1}{2}\mathrm{CA}, \quad \sin\tfrac{1}{2}\mathrm{CZ}\sin\tfrac{1}{2}\mathrm{AB},$$

the sum of any two is equal to the third, the circles will touch.

Let L be one of the limiting points of the coaxal system determined by the circles; then, by the previous Article,

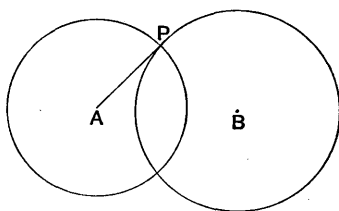
$$\frac{\sin\frac{1}{2}AX}{\sin\frac{1}{2}AL}=\frac{\sin\frac{1}{2}BY}{\sin\frac{1}{2}BL}=\frac{\sin\frac{1}{2}CZ}{\sin\frac{1}{2}CL}. \quad\text{.................(21)}$$

Therefore the specified condition becomes that, of the three products

$$\sin\tfrac{1}{2}AL\sin\tfrac{1}{2}BC, \quad \sin\tfrac{1}{2}BL\sin\tfrac{1}{2}CA, \quad \sin\tfrac{1}{2}CL\sin\tfrac{1}{2}AB,$$

the sum of two is equal to the third. But, when this is the case, we know by Art. 172 that L lies on the circle ABC; that is to say, the limiting point lies on one of the circles; therefore the circles touch.

179. If two small circles cut orthogonally, the plane of either passes through the vertex of the cone that touches the sphere along the circumference of the other.

Let A, B be the poles of the two circles, P one of their points of intersection.

Since the arc AP and the circle B touch one another at P, they have the same tangent line at that point. And this line lies in the plane of the circle B, and passes through the vertex of the cone that touches the sphere along the circumference of A. Therefore the plane of B passes through the vertex of the cone.

From the result just proved we make the following obvious inferences;

(1) If any number of circles on a sphere have a common orthogonal circle, their planes pass through a common point.

(2) If any number of circles on a sphere have two common orthogonal circles, their planes pass through two common points; that is to say, they pass through a common straight line. Thus, also, the planes pass through an infinite number of common points; and so circles which have two common orthogonal circles have an infinite number of them.

Now the circles of a coaxal system have an infinite number of common orthogonal circles, having their poles on the radical circle. Hence the planes of the circles of a coaxal system pass through a common straight line, which lies in the plane of the radical circle.

This property of a coaxal system is given by Dr. CASEY as the definition of such a system, and he deduces all other properties from it. See his *Spherical Trigonometry*, Chap. VI.

180. To find how many great circles can be drawn to touch two given small circles.

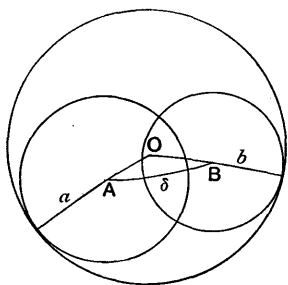
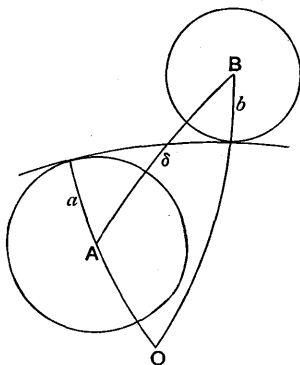



FIG. 1. FIG. 2.

Let A, B be the poles and a, b the radii of two small circles. If possible let these both be touched by the same great circle, whose pole is O. We shall denote the distance AB by δ, and

when the radii a, b are unequal we shall suppose a to be the greater.

If the small circles lie on the same side of the great circle, as in Figure 1, the great circle may, from analogy with plane geometry, be called an external spherical common tangent of the small circles. Clearly when one such tangent exists there is also another, and the two are symmetrically situated with respect to the arc AB.

Now referring to Figure 1 we see that, in the triangle AOB, $AO = \frac{1}{2}\pi - a$, $BO = \frac{1}{2}\pi - b$, and AB is what we have called δ. But two sides of a triangle are together greater than the third, and hence the triangle AOB is only possible provided

$$AO + OB > AB, \text{ and } AO + AB > OB;$$

that is, provided $\pi > a + b + \delta$, and $\delta > a - b$.

Thus if δ lie between $a - b$ and $\pi - (a + b)$ there are two external common tangents, but not otherwise. This result may be stated thus: two given circles have no external common tangents (1) if one lie completely inside the other, (2) if the circles having the same poles, but radii respectively the complements of the original radii, lie completely outside one another. Calling these latter the complementary circles of the original small circles, the result may be stated even more compactly thus. Two circles have or have not external common tangents, according as their complementary circles do or do not intersect.

In the limiting case when δ becomes equal to $a - b$, the two external common tangents coincide, their points of contact with the small circles also coincide, and the figure becomes a great circle and two small circles all having internal contact with one another at the same point.

In the other limiting case, when δ becomes equal to $\pi - (a + b)$, the common tangents again coincide, but their coincident points of contact with one of the small circles do not coincide with their coincident points of contact with the

other. These points of contact are in fact diametrically opposite to one another, and the small circles do not in general touch each other at all; however each has external contact with the circle antipodal to the other; their complementary circles moreover have external contact. This case has no analogue in plane geometry.

If there be a great circle touching two given small circles, and having them on opposite sides of itself, the diagram will be of the same character as Figure 2. The great circle in this case may be called an internal common tangent. The sides of the triangle AOB are now $\frac{1}{2}\pi - a$, $\frac{1}{2}\pi + b$, and δ. In order that the triangle may be possible it is necessary that

$$\text{OA} + \text{AB} > \text{OB}, \text{ and } \text{OA} + \text{OB} > \text{AB},$$

or

$$\delta > a + b, \text{ and } \delta < \pi - (a - b).$$

In the limiting case when δ becomes equal to $a + b$, the internal common tangents coincide, and we have the two small circles touching each other externally and touching the great circle all at the same point.

In the other limiting case, when δ becomes equal to $\pi - (a - b)$, the two internal common tangents again coincide. And now the figure consists of two small circles not meeting one another at all, and touching the same great circle on opposite sides at diametrically opposite points. Either circle has internal contact with the circle antipodal to the other.

EXAMPLES XII.

Proofs of the following theorems are given in GUDERMANN'S *Niedere Sphärik*. They are set here as exercises, as the student who is familiar with the methods of Plane Geometry will have no difficulty in proving them for himself.

1. Two spherical triangles, having the same vertical angle and the same escribed circle opposite to that angle, have equal perimeters.

2. The tangent at A to the circumcircle of a triangle ABC makes with AC and AB angles whose difference is equal to the difference of the angles B and C.

3. If, from a point P outside a small circle, tangent arcs PA, PB and a secant PCD be drawn, and if the tangent arc at C meet PA, PB, and AB in M, N, and U respectively, then

$$\sin UN \sin CM = \sin UM \sin NC.$$

[**Definition.** When this relation holds between four points on a great circle it is said to be divided harmonically, and the four points constitute a *harmonic range.*]

4. If, with the notation of the previous question, AB intersect CD in E, PE is divided harmonically by C and D.

5. Shew also that

$$\sin^2\tfrac{1}{2}APB = \frac{\sin AM}{\sin MP} \cdot \frac{\sin BN}{\sin NP}.$$

6. If AB be a diameter of a small circle, and the tangent arcs at A and B meet any other tangent arc in M and N respectively, then

$$\sin^2\tfrac{1}{2}AB = \tan AM \tan BN.$$

7. ABC is a spherical triangle, and a small circle cuts BC in P and P′, CA in Q and Q′, AB in R and R′; then

$$\frac{\sin AQ \sin AQ'}{\cos^2\frac{1}{2}QQ'} = \frac{\sin AR \sin AR'}{\cos^2\frac{1}{2}RR'},$$

and
$$\frac{\sin BP \sin BP'}{\sin CP \sin CP'} \cdot \frac{\sin CQ \sin CQ'}{\sin AQ \sin AQ'} \cdot \frac{\sin AR \sin AR'}{\sin BR \sin BR'} = 1.$$

8. The opposite sides of a spherical hexagon inscribed in a small circle intersect in points which all lie on the same great circle.

9. If AP, AP′, BQ, BQ′, CR, CR′ be tangents from the corners ABC of a spherical triangle to a small circle,

$$\frac{\sin BAP \sin BAP'}{\sin CAP \sin CAP'} \cdot \frac{\sin ACR \sin ACR'}{\sin BCR \sin BCR'} \cdot \frac{\sin CBQ \sin CBQ'}{\sin ABQ \sin ABQ'} = 1.$$

10. If four great circles through a point O intersect two other great circles in A, B, C, D and A′, B′, C′, D′ respectively,

$$\frac{\sin AB \sin CD}{\sin AC \sin BD} = \frac{\sin AOB \sin COD}{\sin AOC \sin BOD} = \frac{\sin A'B' \sin C'D'}{\sin A'C' \sin B'D'}.$$

11. If ABCD be a harmonic range on a great circle, and M the mid point of the arc AC,

$$\tan^2 MC = \tan MB \,.\, \tan MD,$$

and
$$\frac{\sin 2MB}{\sin 2MD} = \left(\frac{\sin AB}{\sin AD}\right)^2.$$

12. The three diagonals of a complete spherical quadrilateral divide one another harmonically.

13. If O, A, B, C, D be five points on the same small circle

$$\frac{\sin\frac{1}{2}AB \sin\frac{1}{2}CD}{\sin\frac{1}{2}BC \sin\frac{1}{2}AD} = \frac{\sin AOB \sin COD}{\sin BOC \sin AOD}.$$

14. The diagonals AC, BD of a spherical quadrilateral, inscribed in a circle, intersect in E. Shew that

$$\frac{\sin AE}{\sin EC} = \frac{\sin\frac{1}{2}AB \sin\frac{1}{2}AD}{\sin\frac{1}{2}BC \sin\frac{1}{2}DC},$$

and that $$\frac{\sin\frac{1}{2}AC}{\sin\frac{1}{2}BD} = \frac{\sin\frac{1}{2}AB \sin\frac{1}{2}AD + \sin\frac{1}{2}BC \sin\frac{1}{2}DC}{\sin\frac{1}{2}AB \sin\frac{1}{2}BC + \sin\frac{1}{2}AD \sin\frac{1}{2}DC}.$$

15. If ABCDEF be a spherical hexagon inscribed in a circle, and if AD, BE, CF meet in a point, then

$$\sin\tfrac{1}{2}AB \sin\tfrac{1}{2}CD \sin\tfrac{1}{2}EF = \sin\tfrac{1}{2}BC \sin\tfrac{1}{2}DE \sin\tfrac{1}{2}FA.$$

16. If two circles cut orthogonally, any great circle through the centre of either is cut by the circumferences *semi-harmonically*; that is, if PRQS be the range,

$$\sin\tfrac{1}{2}PR \sin\tfrac{1}{2}QS = \sin\tfrac{1}{2}QR \sin\tfrac{1}{2}PS.$$

17. A circle, whose pole is P, touches three circles A, B, C all externally; and another circle whose pole is Q touches A, B, C all internally. Shew that the arc joining PQ passes through the point of concurrence of the radical great circles of A, B, C taken in pairs.

18. If two circles touch two other circles, a centre of similitude of one pair lies on the radical circle of the other pair.

19. Two circles whose radii are $\cot^{-1}\alpha$ and $\cot^{-1}\beta$ touch externally. Shew that the angle between their common tangents is

$$2\cos^{-1}[2\sqrt{\alpha\beta - 1}/(\alpha+\beta)].$$

CHAPTER X.

ON THE DUALITY OF THEOREMS RELATING TO GREAT AND SMALL CIRCLES ON A SPHERE.

181. The relation between a great circle and its pole is somewhat analogous to that which subsists in plane geometry between a straight line and its pole with respect to a circle. The former relation is really a much simpler one than the latter, and from it a method may be developed which is easier and more general in its application than the method of reciprocal polars.

182. A great circle has two poles, and it is sometimes necessary to distinguish between the two. For this purpose it is usual to regard the great circle as traced by a point moving in one direction or the other; the choice of direction is arbitrary, but once made it must be adhered to. If we imagine a person to walk on the outside of the sphere along the great circle in the chosen direction, that pole which lies to his left as he walks is called the left-hand pole of the circle, the other the right-hand pole.* Thus, on the earth, if the direction arbitrarily assigned to the great circle of the equator be from west to east, the left-hand pole will be the north pole of the earth, the right-hand pole the south pole. When we speak of the pole of a great circle it will be understood that

* Attention was drawn to this important distinction by GAUSS (*Disqu. gen. circa superficies curvas*, 2, vi.). Cf. SCHULZ, *Sphärik*, I, 12, and MÖBIUS, *Anal. Sphärik*, 16 and 18.

the left-hand pole is meant. The direction assigned to the circle will be indicated when necessary by an arrow.

183. In order that the poles of the sides of a spherical triangle should be the corners of the polar triangle as defined in Art. 25, it is necessary that the directions assigned to the sides should be such that, following them, a person would travel round the triangle keeping its area always on his left.

184. The poles of small circles may be distinguished in a similar manner; but when no special direction is assigned to a small circle we shall understand by its pole that pole which is nearer to its circumference.

185. If we take any point on the sphere, there is a great circle of which, the proper direction having been assigned, it is the left-hand pole. This great circle, with this direction assigned to it, will, for the purpose of the present discussion, be called the *polar great circle* of the point, or briefly its *polar circle.*

186. The angle between two great circles is the angle between the positive directions of their arcs at the point of intersection.*

187. The following properties of points and their polar circles are now readily verified:

(1) If three points lie on a great circle, their polar circles pass through a point.

(2) The (angular) distance between two points is equal to the angle of intersection of their polar circles.

(3) If A, B be two points, and a, b their polar circles, the intersections of a and b are the poles of the great circles joining A and B. That intersection, at which the less angle between the positive directions of the arcs is described by a

* Gauss, *loc. cit.*

counter-clockwise * rotation from a to b, is the pole of the circle AB taken with its positive direction from A to B along the less arc.

(4) If a moving point trace out on the sphere any curve P, its polar constantly touches another curve Q. And since the join of two points on P has for its pole the intersection of the two corresponding spherical tangents to Q, we find, on making the points on P coincide, that the polar circle of any point on Q touches the curve P. Thus the relation between P and Q is a reciprocal one, and either curve may be called the *reciprocal* of the other.

There are, indeed, two curves, antipodes of one another, both of which are touched by the polar circle of the point that traces the curve P. Round one of these the point of contact moves in the same sense (clockwise or counter-clockwise †) in which the tracing point describes P; round the other, it moves in the opposite sense. It is the former of these two which is called the reciprocal of the curve P; when oval or circular, its concavity is towards the left of the great circle which envelopes it.

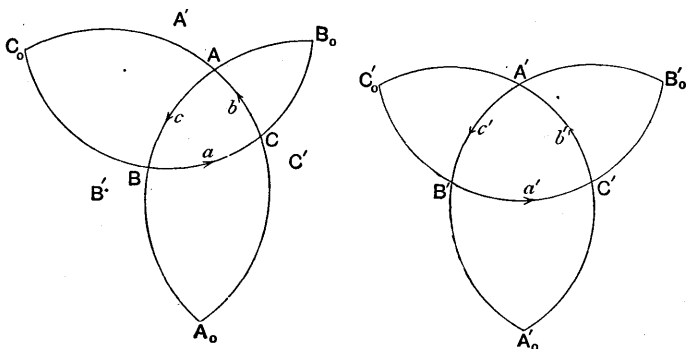
(5) The reciprocal of a small circle is another small circle. The two circles have the same pole, and the sum of their spherical radii is a right angle. ‡

188. Notation. If we denote by a small letter, such as a, a great circle or a part of it, with a certain direction assigned

* If we suppose a watch to be laid on the outside of the sphere, with its face outwards, the sense in which its hands would rotate is that which we shall speak of as the *clockwise* sense; the opposite sense we shall call counter-clockwise.

† These terms could hardly be said to have a definite meaning without further explanation, if applied to the description of large closed curves of such shape as not to be included in one hemisphere; but when applied to small circles they are free from ambiguity.

‡ SCHULZ, *Sphärik*, I, § 44.

as positive, it will be convenient to denote by $-a$ the same great circle or part of it, with the reverse direction assigned as positive. It will also be convenient to denote by (a, b), or sometimes more briefly by (ab), the angle between the arcs a and b, measured from a to b in the counter-clockwise sense.

189. Let us consider now the relations that exist between the figure formed by three great circles and its reciprocal figure.

Let a, b, c be three great circles having directions on them assigned as positive; these directions are indicated in the figure by arrows. Let ABC be the triangle which they form, and A_0BC, AB_0C, ABC_0 the colunar triangles.

Let the poles of a, b, c be A′, B′, C′, and let these be joined by great circles a', b', c', forming the polar triangle A′B′C′ and its colunar triangles $A'_0B'C'$, $A'B'_0C'$, $A'B'C'_0$. Then A, B, C are the poles of a', b', c' when the directions assigned are those indicated by arrows in the right-hand diagram.

It is to be noticed that *the triangle formed by three great circles is that one of the various curvilinear triangular areas which lies to the left of all its sides.* The present figures are so drawn that neither the triangle ABC nor its polar triangle A′B′C′ has

reëntrant angles; but the exclusion of reëntrant angles is not at all necessary in the present theory.

190. The first thing that strikes us in the present figures is that the angle (ab) is not the angle BCA of the triangle, but its supplement B_0CA. And as the angle between two great circles is equal to the angle between their poles, we have

$$\pi - C = (ab) = A'B',$$

and reciprocally

$$\pi - C' = (a'b') = AB,$$

which are the fundamental properties of polar or supplemental triangles.

191. We have seen that if any point lie on a given small circle, its polar great circle touches (on its left-hand side) a certain other small circle. If, therefore, we assume the existence of the circum-circle of the triangle ABC, we can deduce from it the existence of a small circle touching the arcs a', b', c', and lying to the left of each of them; this is the inscribed circle of the triangle $A'B'C'$. Hence the following theorem:

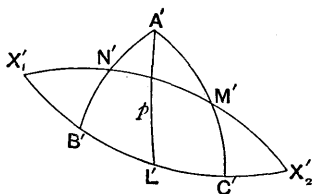
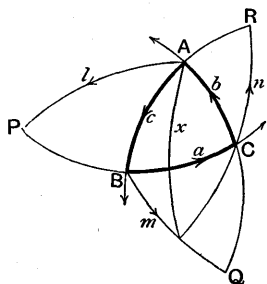
The pole of the circumscribed circle of a triangle coincides with the pole of the inscribed circle of the supplemental triangle; and the spherical radii of the two circles are complementary.

This result enables us to deduce the value of $\cot r$ from that of $\tan R$ by substituting the elements of the polar triangle.

192. If we reverse the direction assigned to the great circle a, that is to say if we consider the great circle $-a$, we find that it forms with the great circles b and c the triangle A_0BC. The pole of $-a$ is A'_0, and therefore A_0BC, $A'_0B'C'$ are supplemental triangles. Hence the following result:

The pole of the circle circumscribed to one of the colunar triangles of a given triangle coincides with the pole of the circle inscribed in the corresponding colunar triangle of the supplemental triangle; and the spherical radii of the two circles are complementary.

193. If A, P, B be three points on a great circle, and a', p', b' their polar great circles, $AP = (a'p')$, $PB = (p'b')$. Hence if P be the mid point of the arc AB, p is the great circle which bisects internally the angle between the positive directions of the circles a, b. The arc drawn through P at right angles to AB has for poles the points on p distant a quadrant from the intersections of a and b.

194. Let L′, M′, N′ be the mid points of the sides of the triangle A′B′C′; the polar circles of these points are the great circles l, m, n, which bisect the external angles of the triangle ABC. The intersections of l, m, n with a, b, c respectively (say P, Q, R and their antipodal points), are poles of the medians A′L′, B′M′, C′M′, (or p, q, r). Now p, q, r are concurrent (Art. 160), and therefore P, Q, R lie on a great circle.

Again, the arcs drawn through L′, M′, N′ perpendicular to a', b', c', meet in a point, namely the pole of the circum-circle of A′B′C′. Hence the points on l, m, n distant a quadrant from A, B, C respectively, lie on a great circle whose pole is the pole of the inscribed circle of ABC.

Again, the arc x joining the intersection of b and c to that of m and n is the internal bisector of the angle A, and therefore is at right angles to l; hence X'_1, X'_2, the intersections of M′N′ and B′C′, are distant a quadrant from L′.

And since the three arcs such as x meet in a point, namely the pole of the inscribed circle of ABC, the six points such as X'_1 lie on a great circle having the same pole as the circle circumscribed to A′B′C′.

195. LEXELL'S locus may be derived as follows. Since the length of the spherical tangent from A to the circle inscribed in the colunar triangle A_0BC is s the semiperimeter, it follows that when one angle of a triangle is given in position, and the perimeter is given, the opposite side constantly touches two small circles, namely the inscribed circle of the colunar triangle, and the circle antipodal to it. Of these it is the latter that lies to the left of a and is therefore the one to be used in the present connection. Now, reciprocating the theorem, we find that if one side and the sum of the angles of a triangle be given, the opposite vertex lies constantly on a circle which is antipodal to the circum-circle of the colunar triangle (Art. 192), and therefore passes through the points diametrically opposite to the extremities of the given base. This is LEXELL'S locus.*

196. The following theorems are reciprocals of one another.

(1) If the magnitude and position of the base BC of a spherical triangle be given, and the ratio of $\sin\frac{1}{2}b$ to $\sin\frac{1}{2}c$, the locus of A is a small circle (Art. 166).

(1′) If the magnitude and position of the angle A of a spherical triangle be given, and the ratio of $\cos\frac{1}{2}B$ to $\cos\frac{1}{2}C$, the envelope of BC is a small circle.

(2) The sum of one pair of opposite angles of a four-sided figure inscribed in a small circle is equal to the sum of the other pair (Art. 168).

(2′) The sum of one pair of opposite sides of a four-sided figure circumscribed to a small circle is equal to the sum of the other pair.

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* The theorem reciprocal to LEXELL'S theorem, and from which we here derive the latter, was first published by SORLIN, GERGONNE'S *Annales de Mathématiques*, XV, p. 302.

(3) If ABCD be a quadrangle inscribed in a small circle, two of the products

$$\sin\tfrac{1}{2}\mathsf{AB}\sin\tfrac{1}{2}\mathsf{CD},$$
$$\sin\tfrac{1}{2}\mathsf{AC}\sin\tfrac{1}{2}\mathsf{BD},$$
$$\sin\tfrac{1}{2}\mathsf{BC}\sin\tfrac{1}{2}\mathsf{AD},$$

are together equal to the third (Art. 172).

(3′) If a, b, c, d be four great circles all touching the same small circle, and if the directions assigned be such that the small circle lies to the left of each of the great circles, then two of the products

$$\sin\tfrac{1}{2}(ab)\sin\tfrac{1}{2}(cd),$$
$$\sin\tfrac{1}{2}(ac)\sin\tfrac{1}{2}(bd),$$
$$\sin\tfrac{1}{2}(bc)\sin\tfrac{1}{2}(ad),$$

are together equal to the third.

(4) Given two sides (together not greater than 180°) of a spherical triangle, the area is maximum when the third side is the diameter of the circum-circle (Arts. 155, 156).

(4′) Given two angles (together greater than 180°) of a spherical triangle, the perimeter is least when the third angle is equal to the diameter of the inscribed circle.*

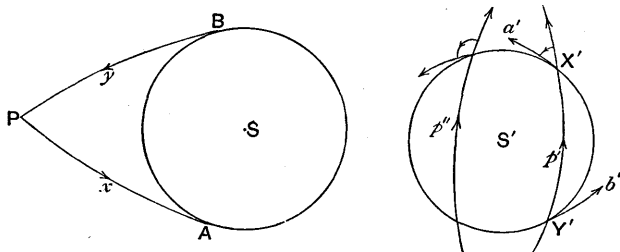
197. Before passing to further theorems relating to small circles, it will be well to consider briefly the reciprocation of chords and tangents to small circles.

If through a point P we draw a great circle x to cut a given small circle S in points A and B, then the reciprocals of the various elements of the figure are as follows. Corresponding to S we get another small circle S′. The reciprocal of P is a great circle p', that of x is a point X′ on the great circle p'; the reciprocals of A and B are great circles a', b' drawn from X′ to touch the circle S′. And so PA $=(p'a')$, PB $=(p'b')$, AB $=(a'b')$. While if O be the common pole of S and S′, PO equals the complement of the distance of p' from O. These relations are exemplified in the following pair of theorems.

If a variable arc of a great circle, passing through a fixed point P, cut a given small circle in A and B, the product $\tan\frac{1}{2}\mathsf{PA}\tan\frac{1}{2}\mathsf{PB}$ has a constant value (Art. 170).

If through a variable point P on a fixed great circle PN the spherical tangents PE, FP be drawn to a given small circle, the product $\tan\frac{1}{2}\mathsf{NPE}\cot\frac{1}{2}\mathsf{NPF}$ has a constant value (Art. 171).

* BALTZER, *Sterometrie*, § IV, 17.

198. If through a point P great circles x and y be drawn to touch a small circle S at the points A and B respectively, the reciprocal diagram consists of a great circle p' cutting a small circle S′ in points X′ and Y′, at which points the spherical tangents to the circle S′ are great circles a' and b' respectively. And it is now seen that the arc PA is equal to the angle $(p'a')$, that is, the angle between the great circle p' and the tangent at X′ to the small circle S′; it is, in fact, the angle at which the arc p' cuts the small circle. But in estimating this angle it is most important to bear in mind that the positive direction of the tangent arc a' is such that the small circle lies

on its left-hand side. We shall define the angle as a rotation *from* p' *to* a', and reckon it positive when the rotation is in the counter-clockwise sense; with this convention it appears that, in the figure as drawn above, the angle of intersection of the great and small circles is positive at X′, negative at Y′, just as the arc PA is in the positive direction of the great circle x, while PB is in the negative direction of the great circle y. Thus the convention may be formulated as follows. Let a point travelling along a great circle p' enter a small circle S′ at the point Y′, and emerge from it at the point X′; the angle of intersection of p' and the small circle is the amount of counter-clockwise rotation from the great circle to the tangent to the small one at the point of emergence.

In the present figures P is represented as distant from the common pole of S and S′ by less than a quadrant; this pole accordingly lies to the left of p', and the angle of intersection is acute. But if P were at a greater distance than a quadrant from the pole of the small circles, the pole would lie to the right of p' (as, for instance, p'' in the figure), and the angle of intersection would be obtuse.

199. The angle of intersection of two small circles is the angle between the tangent great circles at one of their common points, the direction assigned to each tangent being such that the corresponding small circle lies to the left. Thus, for example, the angle of intersection of two small circles having internal contact is zero; and the angle of intersection of two small circles having external contact is π.

It will readily be seen that the angle of intersection of two small circles is equal to the length of the external common tangent of their reciprocals. Thus if two circles cut orthogonally the common tangent of their reciprocals is a quadrant. If two circles touch internally, their reciprocals touch internally. If two circles touch externally, the length of the common tangent to their reciprocals is a semicircle; so that the reciprocals touch the same great circle, on the same side of it, at diametrically opposite points.

200. All the propositions given above (Arts. 174-178), for circles of a coaxal system, may now be reciprocated, and we thus obtain the properties of another system of small circles on the sphere. We shall call this new system of circles a *colunar* system, since all the circles belonging to it have a common pair of spherical tangents, real or imaginary, and may therefore be said to be inscribed in the same lune. In the left-hand column are given the propositions already proved and definitions already made; in the right-hand column are the reciprocal propositions and definitions.

I. The locus of points from which the spherical tangents drawn to two given small circles are equal is a great circle (Art. 174).

II. This great circle is called the *radical circle* of the two given circles.

III. The radical circle is orthogonal to the arc joining the poles of the two circles.

IV. If A, B be the poles, a, b the radii of the circles, and O the intersection of the radical circle with AB,

$$\frac{\cos \mathrm{OA}}{\cos a}=\frac{\cos \mathrm{OB}}{\cos b}.$$

V. O is called the *axial centre* of the two circles.

VI. A system of circles such that all pairs of circles of the system have the same radical circle is called a *coaxal* system of circles (Arts. 175-178).

VII. If A, B, C, etc. be the poles, a, b, c, etc. the radii of the circles of a coaxal system, then

(1) A, B, C, etc. must be on the same great circle (the circle of centres).

(2) There is a certain point O (the axial centre) on this great circle, such that

$$\frac{\cos \mathrm{OA}}{\cos a}=\frac{\cos \mathrm{OB}}{\cos b}=\frac{\cos \mathrm{OC}}{\cos c}=\ldots=k, \text{ say.}$$

(3) The tangents to all circles of the system, from any point P on the radical circle, are equal.

I′. All great circles which cut two given small circles at the same angle pass through a certain pair of diametrically opposite points.

II′. These points are called the *external centres of similitude* of the two given circles.

III′. The centres of similitude lie on the great circle joining the poles of the two circles.

IV′. If A, B be the poles, a, b the radii of the circles, and O the point on the great circle AB midway between the centres of similitude,

$$\frac{\cos \mathrm{OA}}{\sin a}=\frac{\cos \mathrm{OB}}{\sin b}.$$

V′. O is called the *lunar centre* of the two circles.

VI′. A system of circles such that all pairs of circles of the system have the same external centres of similitude is called a *colunar* system of circles.

VII′. If A, B, C etc. be the poles, a, b, c, etc. the radii of the circles of a colunar system, then

(1′) A, B, C, etc. must be on the same great circle (the circle of centres).

(2′) There is a certain point O (the lunar centre) on this great circle, such that

$$\frac{\cos \mathrm{OA}}{\sin a}=\frac{\cos \mathrm{OB}}{\sin b}=\frac{\cos \mathrm{OC}}{\sin c}=\ldots=k, \text{ say.}$$

(3′) Any great circle p', which passes through the centres of similitude, cuts all circles of the system at the same angle.

(4) A circle with such point P as pole, and the common length of tangent as spherical radius, cuts all circles of the system orthogonally.

(4′) The length of the common tangent of any circle of the system and a circle, whose pole is the pole of such great circle p' and radius the complement of the common angle of intersection of p' with the circles, is a quadrant.

(5) When the quantity k, defined above, is greater than unity, all the circles of the system pass through two real common points.

(5′) When the quantity k, defined above, is greater than unity, all the circles of the system have the same two real common tangent great circles.

(6) When k is less than unity no two of the circles intersect.

(6′) When k is less than unity no two of the circles have an external common tangent great circle (Art. 180).

(7) When k is less than unity there are two point circles of the system, that is, circles of zero radius. These are called the *limiting points*; they are symmetrically situated with respect to the axial centre.

(7′) When k is less than unity there are two great circles belonging to the system. These are called the *limiting circles*; they are symmetrically situated with respect to the lunar centre.

(8) The radical circle is itself a circle of the coaxal system.

(8′) There are two point circles of a colunar system, namely the centres of similitude.

(9) When two circles touch, internally or externally, the axial centre and the limiting points coincide at the point of contact, and all the circles of the system touch one another at that point, internally or externally. This is the case when $k=1$; it is the transition case between (5) and (6).

(9′) When two circles touch the same great circle, either at the same point or at points diametrically opposite to one another, the limiting circles of the system coincide with one another and with this great circle, and the lunar centre is its pole. The system then consists of circles each of which touches the limiting circle at one or other of two diametrically opposite points, the centres of similitude. Thus each circle of the system touches each other internally, or else the circle antipodal to it externally. This is the case when $k=1$; it is the transition case between (5′) and (6′).

VIII. The radical circles of three small circles, taken in pairs, meet in a point.

IX. A circle may be described to cut three given circles orthogonally.

X. A circle can be described such that the tangents to it from three given points shall have given lengths.

XI. If A, B, C be three circles of a coaxal system, and from a variable point P on C tangents PL, PM be drawn to A and B, then

$$\frac{\sin^2 \frac{1}{2} PL}{\sin^2 \frac{1}{2} PM} = \frac{\sin AC}{\sin BC} \cdot \frac{\cos b}{\cos a},$$

and is therefore constant wherever P may be on the circle C.

XII. If A, B, C be three points on a circle, AX, BY, CZ spherical tangents to another circle, then if the sum of any two of the products

$$\sin \tfrac{1}{2} AX \sin \tfrac{1}{2} BC, \quad \sin \tfrac{1}{2} BY \sin \tfrac{1}{2} CA, \quad \sin \tfrac{1}{2} CZ \sin \tfrac{1}{2} AB,$$

is equal to the third, the circles touch.

VIII′. The external centres of similitude of three small circles, taken in pairs, lie on a great circle.

IX′. A circle may be described whose distance, measured along an external common tangent, from each of three given small circles shall be a quadrant.

X′. A circle can be described to cut three given great circles at given angles.

XI′. If A, B, C be three circles of a colunar system, and if a variable tangent to the circle C cut the circles A and B at angles α and β, then

$$\frac{\sin^2 \frac{1}{2} \alpha}{\sin^2 \frac{1}{2} \beta} = \frac{\sin AC}{\sin BC} \cdot \frac{\sin b}{\sin a},$$

and is therefore constant, whatever tangent we take to the circle C.

XII′. If a, b, c be three great circles touching a small circle, and cutting another small circle at angles α, β, γ respectively, then if the sum of any two of the products

$$\sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}(bc), \quad \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}(ca), \quad \sin \tfrac{1}{2}\gamma \sin \tfrac{1}{2}(ab),$$

is equal to the third, either the small circles touch one another internally, or else one touches externally the circle antipodal to the other.

201. It should be noticed that in the present discussion the circles of a coaxal system have tacitly been assumed subject to the restriction that their poles are never distant by more than a quadrant from the axial centre; and therefore also

the circles of a colunar system are confined to *one* of the lunes formed by two great circles. In other words, we have contemplated only positive values of k. For, if we consider the equality $\cos \mathsf{OA} = k \cos a$, where k is positive, we notice that if $\mathsf{OA} > \frac{1}{2}\pi$, $\cos a$ is negative, and therefore $a > \frac{1}{2}\pi$. Thus we get a small circle of radius greater than a quadrant, and therefore one which has (though from a different point of view) been considered already in connexion with its nearer pole. The restriction is unnecessary, but it shortens the discussion. By extending the original definitions, and by assigning an arbitrary direction of rotation to each small circle, instead of adhering, as we virtually have done, to the counter-clockwise sense, a theory could readily be built up which would apply (in the case of colunar systems) to internal as well as external centres of similitude.

It is also to be remarked that we cannot, by making the radius of the sphere infinite, derive from the method of reciprocation here developed for the sphere a correspondingly effective reciprocation of points, lines, and circles on a plane. For on the sphere a diagram and its reciprocal are in general removed from one another by a distance comparable with a quadrant, and therefore will be infinitely far apart on a sphere of infinite radius; consequently on a plane the reciprocal diagrams are infinitely distant from one another. In plane geometry, for example, we have coaxal circles and colunar circles, with properties easily deduced from those enumerated in Art. 200; but there is no obvious method of plane geometry by which the properties of either system can be inferred from those of the other.

CHAPTER XI.

HART'S CIRCLE.

202. Hart's theorem. It is a well-known theorem in Plane Geometry, usually associated with the name of FEUERBACH, that the inscribed and escribed circles of a triangle are all touched by another circle, namely, the Nine Points Circle. An analogous theorem for spherical triangles was discovered by Sir ANDREW HART in 1861, and his demonstration of it will be found in the *Quarterly Journal of Mathematics* for that year.* It is to the effect that the inscribed circles of a spherical triangle and its colunar triangles are all touched by a fourth small circle.

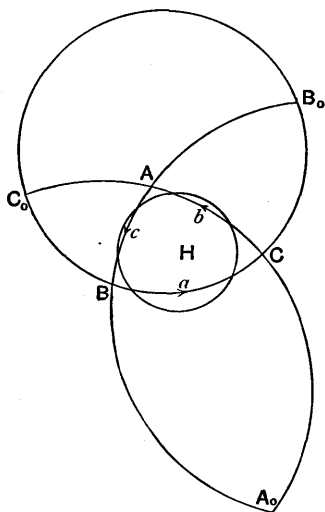
We shall here deduce a proof of the theorem from the properties of a colunar system of circles, making use of the theorem of Art. 200 (XII′).

203. Let ABC be a spherical triangle, in which we shall suppose that A is not less than B, and B not less than C.

We have seen (Art. 200, X′) that it is always possible to describe a small circle to cut the sides a, b, c of the triangle at given angles α, β, γ; and further that, if α, β, γ satisfy the condition that of the three products

$$\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}(b,\ c),\quad \sin\tfrac{1}{2}\beta\sin\tfrac{1}{2}(c,\ a),\quad \sin\tfrac{1}{2}\gamma\sin\tfrac{1}{2}(a,\ b),$$

* *Extension of* TERQUEM'S *Theorem respecting the circle which bisects three sides of a triangle. Quarterly Journal*, Vol. IV, p. 260.

the sum of any two is equal to the third, then the circle which has been described will have either internal contact with the inscribed circle of the triangle, or external contact with the circle antipodal to the inscribed circle (Art. 200, XII′).

Let A_0, B_0, C_0 be the points antipodal to A, B, C.

Consider now the small circle H, which cuts *a*, *b*, and *c* at angles $B-C$, $A-C$, and $A-B$ respectively. Then, since $(b, c)=\pi-A$, the three products are

$$\sin\tfrac{1}{2}(B-C)\cos\tfrac{1}{2}A, \quad \sin\tfrac{1}{2}(A-C)\cos\tfrac{1}{2}B, \quad \sin\tfrac{1}{2}(A-B)\cos\tfrac{1}{2}C;$$

these are respectively equal to

$$-\tfrac{1}{2}\sin(S-B)+\tfrac{1}{2}\sin(S-C), \quad \tfrac{1}{2}\sin(S-C)-\tfrac{1}{2}\sin(S-A),$$
$$-\tfrac{1}{2}\sin(S-A)+\tfrac{1}{2}\sin(S-B),$$

so that the sum of the first and third is equal to the second. Hence the circle H touches either the incircle internally or its antipodal circle externally. Taking the particular case of an equilateral triangle, we find that the circle H and the incircle

coincide. Thus the former of the alternatives is that which applies in the general case.*

204. Next let us consider the triangle AB_0C_0, formed by the great circles a, $-b$, and $-c$. The same circle H cuts these great circles at angles

$$B-C, \quad \pi-(A-C), \quad \text{and} \quad \pi-(A-B)$$

respectively. And therefore the products,

$$\sin\tfrac{1}{2}\alpha \sin\tfrac{1}{2}(-b, -c), \quad \sin\tfrac{1}{2}\beta \sin\tfrac{1}{2}(-c, a), \quad \sin\tfrac{1}{2}\gamma \sin\tfrac{1}{2}(a, -b),$$

are in this instance

$$\sin\tfrac{1}{2}(B-C)\cos\tfrac{1}{2}A, \quad \cos\tfrac{1}{2}(A-C)\sin\tfrac{1}{2}B, \quad \cos\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}C;$$

these are respectively equal to

$$-\tfrac{1}{2}\sin(S-B)+\tfrac{1}{2}\sin(S-C), \quad \tfrac{1}{2}\sin(S-C)+\tfrac{1}{2}\sin(S-A),$$
$$\tfrac{1}{2}\sin(S-A)+\tfrac{1}{2}\sin(S-B),$$

so that the sum of the first and third is equal to the second. Hence the circle H has either internal contact with the circle inscribed in the triangle AB_0C_0, or external contact with the circle antipodal to it, namely the circle inscribed in the triangle A_0BC. Taking the particular case when ABC is equilateral, we see that only the latter alternative is admissible.*

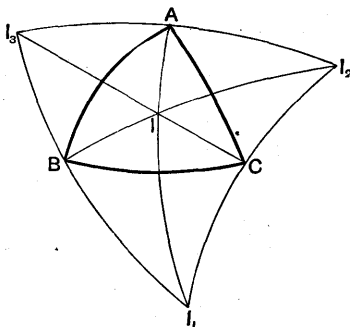
205. Extending similar reasoning to the triangles A_0BC_0 and A_0B_0C, we find that the circle H touches internally the circle inscribed in the triangle ABC, and externally the circles inscribed in its colunar triangles. Thus HART'S theorem is established. The circle H is called *Hart's circle.*

206. Expression for the radius of Hart's circle. The spherical radius of HART'S circle may be found as follows

Let ρ and R be the radii of HART'S circle and the circum circle of the triangle; r, r_1, r_2, r_3 the radii, and I, I_1, I_2, I_3 the

* These appeals to a particular case, though convincing, are somewhat unsatisfactory. The ambiguity has its origin in the imperfect specification of a small circle deliberately adopted in previous chapters. See § 336.

poles, of the inscribed and escribed circles. Then, since the circle H touches these four circles,

$$\mathrm{HI}=\rho-r, \quad \mathrm{HI}_1=\rho+r_1, \quad \mathrm{HI}_2=\rho+r_2, \quad \mathrm{HI}_3=\rho+r_3.$$

Also if 2ν, $2\nu_1$, $2\nu_2$, $2\nu_3$ be the sines of the triangles $\mathrm{I_1I_2I_3}$, $\mathrm{II_2I_3}$, $\mathrm{II_3I_1}$, $\mathrm{II_1I_2}$, we notice that

$$\begin{aligned}\nu_1 : \nu &= \sin \mathrm{AI} : \sin \mathrm{AI}_1 \\ &= \sin r : \sin r_1 ;\end{aligned}$$

hence $$\nu : \nu_1 : \nu_2 : \nu_3 = \frac{1}{\sin r} : \frac{1}{\sin r_1} : \frac{1}{\sin r_2} : \frac{1}{\sin r_3}.$$

Now applying the theorem of Art. 165, we get

$$\nu \cos \mathrm{HI} = \nu_1 \cos \mathrm{HI}_1 + \nu_2 \cos \mathrm{HI}_2 + \nu_3 \cos \mathrm{HI}_3,$$

which is equivalent to

$$\frac{\cos(\rho-r)}{\sin r} = \frac{\cos(\rho+r_1)}{\sin r_1} + \frac{\cos(\rho+r_2)}{\sin r_2} + \frac{\cos(\rho+r_3)}{\sin r_3}.$$

Thus $$\begin{aligned}4 \tan \rho &= \cot r_1 + \cot r_2 + \cot r_3 - \cot r \\ &= \frac{1}{n}\{\sin(s-a)+\sin(s-b)+\sin(s-c)-\sin s\}, \text{ (Arts. 119, 120)} \\ &= 2 \tan \mathrm{R}, \quad \text{(Art. 122)}\end{aligned}$$

and therefore $\tan \rho = \frac{1}{2} \tan \mathrm{R}$.

This result was first given by Dr. SALMON;* the proof of the present Article is that given by Dr. CASEY.†

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* *Quarterly Journal of Mathematics*, 1864, Vol. VI, p. 72.

† *Spherical Trigonometry*, p. 82.

207. Another method of finding the radius of Hart's circle. The value of ρ and the lengths of the arcs AH, BH, CH may be obtained simultaneously by another and more elementary method as follows.

Apply the theorem of Art. 145 firstly to the points I_2, A, I_3, H, and secondly to the points A, I, I_1, H; thus we get

$$\cos(\rho + r_2)\sin AI_3 + \cos(\rho + r_3)\sin AI_2 = \cos AH \sin(AI_2 + AI_3),$$

$$\cos(\rho - r)\sin AI_1 - \cos(\rho + r_1)\sin AI = \cos AH \sin(AI_1 - AI).$$

In these make the following substitutions,

$$\sin AI_2 = \frac{\sin r_2}{\cos \frac{1}{2}A}, \quad \sin AI_3 = \frac{\sin r_3}{\cos \frac{1}{2}A},$$

$$\cos AI_2 = \cos r_2 \cos(s - c), \quad \cos AI_3 = \cos r_3 \cos(s - b),$$

$$\sin AI_1 = \frac{\sin r_1}{\sin \frac{1}{2}A}, \quad \sin AI = \frac{\sin r}{\sin \frac{1}{2}A},$$

$$\cos AI_1 = \cos r_1 \cos s, \quad \cos AI = \cos r \cos(s - a);$$

divide the first equality by $\cos\rho \sin r_2 \sin r_3$, the second by $\cos\rho \sin r_1 \sin r$, and they become

$$\cot r_2 + \cot r_3 - 2\tan\rho$$
$$= \frac{\cos AH}{\cos\rho}\{\cot r_2 \cos(s - c) + \cot r_3 \cos(s - b)\},$$

$$\cot r_1 - \cot r - 2\tan\rho$$
$$= -\frac{\cos AH}{\cos\rho}\{\cot r \cos(s - a) - \cot r_1 \cos s\}.$$

Substitution, on the right-hand sides, of $\frac{\sin s}{n}$, $\frac{\sin(s - a)}{n}$, etc., for $\cot r$, $\cot r_1$, etc., reduces the equations to the still simpler forms

$$\cot r_2 + \cot r_3 - 2\tan\rho = \frac{\cos AH \sin a}{n \cos\rho},$$

$$\cot r_1 - \cot r - 2\tan\rho = -\frac{\cos AH \sin a}{n \cos\rho}.$$

These may be solved simply by addition and subtraction, and the results are

$$\tan\rho = \tfrac{1}{4}(\cot r_1 + \cot r_2 + \cot r_3 - \cot r) = \tfrac{1}{2}\tan \mathrm{R},$$

and
$$\cos \mathrm{AH} = \tfrac{1}{2}\frac{n\cos\rho}{\sin a}\{\cot r_2 + \cot r_3 - \cot r_1 + \cot r\}$$

$$= \tfrac{1}{2}\frac{\cos\rho}{\sin a}\{\sin(s-b) + \sin(s-c) - \sin(s-a) + \sin s\}$$

$$= \frac{\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c}{\cos\tfrac{1}{2}a}\cos\rho.$$

208. Normal coordinates of H. If x, y, z be the lengths of the arcs drawn from H perpendicular to the sides of the triangle ABC, $\sin x$, $\sin y$, and $\sin z$ are now readily evaluated.

For the triangle whose corners are the foot of the perpendicular x, the centre of HART'S circle, and one of its points of intersection with BC, is right-angled and has its angle opposite to x equal to the complement of B – C.

Hence
$$\sin\rho\cos(\mathrm{B}-\mathrm{C}) = \sin x \quad \text{(Art. 73)},$$

which determines $\sin x$, since ρ is now, by the preceding Article, a known quantity.

209. Let O be the point of concurrence of the altitudes of the triangle, G that of the medians, and H the centre of HART'S circle. We shall shew that O, H, and G lie on a great circle.

Let the perpendiculars drawn to the sides of the triangle from O be (x_1, y_1, z_1), those from H (x, y, z), and those from G (x_2, y_2, z_2). Then, by Arts. 208 and 162,

$$\frac{\sin x}{\cos(\mathrm{B}-\mathrm{C})} = \frac{\sin y}{\cos(\mathrm{C}-\mathrm{A})} = \frac{\sin z}{\cos(\mathrm{A}-\mathrm{B})},$$

$$\frac{\sin x_1}{\cos \mathrm{B}\cos \mathrm{C}} = \frac{\sin y_1}{\cos \mathrm{C}\cos \mathrm{A}} = \frac{\sin z_1}{\cos \mathrm{A}\cos \mathrm{B}},$$

$$\frac{\sin x_2}{\sin \mathrm{B}\sin \mathrm{C}} = \frac{\sin y_2}{\sin \mathrm{C}\sin \mathrm{A}} = \frac{\sin z_2}{\sin \mathrm{A}\sin \mathrm{B}}.$$

Hence it follows that

$$\sin x = t_1 \sin x_1 + t_2 \sin x_2,$$
$$\sin y = t_1 \sin y_1 + t_2 \sin y_2,$$
$$\sin z = t_1 \sin z_1 + t_2 \sin z_2,$$

where t_1 and t_2 are certain quantities the values of which are not required for our purpose.

Therefore, by Art. 146, a certain point *in the same great circle* as O and G is at the perpendicular distances x, y, z from the sides a, b, c respectively of the spherical triangle; and hence this point must be the point O.

210. To determine the positions of the points of intersection of Hart's circle with the sides of the triangle.

Suppose that it intersects the side AB at points distant λ and μ respectively from A.

Then by Art. 170 we have

$$\tan \tfrac{1}{2}\lambda \tan \tfrac{1}{2}\mu = \frac{\cos \rho - \cos \mathrm{AH}}{\cos \rho + \cos \mathrm{AH}} = \frac{\cos \frac{1}{2}a - \cos \frac{1}{2}b \cos \frac{1}{2}c}{\cos \frac{1}{2}a + \cos \frac{1}{2}b \cos \frac{1}{2}c}.$$

In the same way we must have by symmetry

$$\tan \tfrac{1}{2}(c-\lambda) \tan \tfrac{1}{2}(c-\mu) = \frac{\cos \frac{1}{2}b - \cos \frac{1}{2}c \cos \frac{1}{2}a}{\cos \frac{1}{2}b + \cos \frac{1}{2}c \cos \frac{1}{2}a}.$$

Substituting in the second of these the value of $\tan \frac{1}{2}\lambda \tan \frac{1}{2}\mu$ given by the first, we obtain

$$\tan \tfrac{1}{2}\lambda + \tan \tfrac{1}{2}\mu = \frac{\cos^2 \frac{1}{2}a - \cos^2 \frac{1}{2}b \cos^2 \frac{1}{2}c + \cos^2 \frac{1}{2}b \sin^2 \frac{1}{2}c}{\cos \frac{1}{2}b \cos \frac{1}{2}c \,(\cos \frac{1}{2}a + \cos \frac{1}{2}b \cos \frac{1}{2}c)}$$

$$= \frac{\cos \frac{1}{2}a - \cos \frac{1}{2}b \cos \frac{1}{2}c}{\cos \frac{1}{2}b \sin \frac{1}{2}c} + \frac{\cos \frac{1}{2}b \sin \frac{1}{2}c}{\cos \frac{1}{2}a + \cos \frac{1}{2}b \cos \frac{1}{2}c}.$$

From this and the first equality we see that

$$\tan \tfrac{1}{2}\lambda = \frac{\cos \frac{1}{2}a - \cos \frac{1}{2}b \cos \frac{1}{2}c}{\cos \frac{1}{2}b \sin \frac{1}{2}c}, \quad \tan \tfrac{1}{2}\mu = \frac{\cos \frac{1}{2}b \sin \frac{1}{2}c}{\cos \frac{1}{2}a + \cos \frac{1}{2}b \cos \frac{1}{2}c}.$$

211. If U_3, V_3 be the points defined by λ and μ respectively, and U_1, V_1, U_2, V_2 the corresponding points on the sides BC and CA, it is readily verified that the arcs AU_1, BU_2, CU_3 are concurrent in the point whose normal coordinates are proportional to

$$\sec \tfrac{1}{2}(\tfrac{1}{2}\pi + 2A - S), \quad \sec \tfrac{1}{2}(\tfrac{1}{2}\pi + 2B - S), \quad \sec \tfrac{1}{2}(\tfrac{1}{2}\pi + 2C - S),$$

and the arcs AV_1, BV_2, CV_3 in the point whose normal coordinates are proportional to

$$\operatorname{cosec} \tfrac{1}{2}(\tfrac{1}{2}\pi + 2A - S), \quad \operatorname{cosec} \tfrac{1}{2}(\tfrac{1}{2}\pi + 2B - S), \quad \operatorname{cosec} \tfrac{1}{2}(\tfrac{1}{2}\pi + 2C - S).$$

In the case of a triangle on a sphere of infinite radius, that is, a plane triangle, $S = \frac{1}{2}\pi$, and the normal (*i.e.* trilinear) coordinates of these two points are proportional to

$$\sec A, \quad \sec B, \quad \sec C,$$

and

$$\operatorname{cosec} A, \quad \operatorname{cosec} B, \quad \operatorname{cosec} C,$$

respectively. The former point is then the orthocentre, and the latter the centre of gravity of the plane triangle.

Thus when the triangle becomes plane, the points U_1, U_2, U_3 become the feet of the perpendiculars from opposite corners, and the points V_1, V_2, V_3 the mid points of the sides; these six points, of course, lie on the Nine Points Circle.

Also in the case of a plane triangle the formula $\tan \rho = \frac{1}{2} \tan R$ reduces to $\rho = \frac{1}{2}R$, a known property of the Nine Points Circle. These facts exemplify the remarkable analogy between HART'S circle and the Nine Points Circle.

EXAMPLES XIII.

1. From the angle C of a spherical triangle a perpendicular is drawn to the arc which joins the middle points of the sides a and b: shew that this perpendicular makes an angle $S - B$ with the side a, and an angle $S - A$ with the side b.

2. From each angle of a spherical triangle a perpendicular is drawn to the arc which joins the middle points of the adjacent sides: shew

that these perpendiculars meet at a point, and that if x, y, z are the perpendiculars from this point on the sides a, b, c respectively,

$$\frac{\sin x}{\sin(\mathrm{S}-\mathrm{B})\sin(\mathrm{S}-\mathrm{C})}=\frac{\sin y}{\sin(\mathrm{S}-\mathrm{C})\sin(\mathrm{S}-\mathrm{A})}=\frac{\sin z}{\sin(\mathrm{S}-\mathrm{A})\sin(\mathrm{S}-\mathrm{B})}.$$

3. Through each angle of a spherical triangle an arc is drawn to make the same angle with one side as the perpendicular on the base makes with the other side: shew that these arcs meet at a point; and that if x, y, z are the perpendiculars from this point on the sides a, b, c respectively,

$$\frac{\sin x}{\cos \mathrm{A}}=\frac{\sin y}{\cos \mathrm{B}}=\frac{\sin z}{\cos \mathrm{C}}.$$

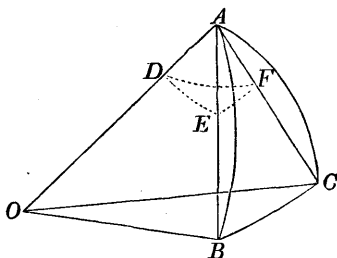
4. Shew that the points determined in Examples 2 and 3, and the pole of HART's circle, are on a great circle.

State the corresponding theorem in Plane Geometry.

CHAPTER XII.

ON CERTAIN APPROXIMATE FORMULAE.

212. We shall now investigate certain approximate formulae which are often useful in calculating spherical triangles when the radius of the sphere is large compared with the lengths of the sides of the triangles.

213. *Given two sides and the included angle of a spherical triangle, to find the angle between the chords of these sides.*

Let AB, AC be the two sides of the triangle ABC; let O be the centre of the sphere. Describe a sphere round A as a centre, and suppose it to meet AO, AB, AC at D, E, F respectively. Then the angle EDF is the inclination of the planes OAB, OAC, and is therefore equal to A. From the spherical triangle DEF

$$\cos EF = \cos DE \cos DF + \sin DE \sin DF \cos A;$$

and $\qquad DE = \frac{1}{2}(\pi - c), \quad DF = \frac{1}{2}(\pi - b)$;

therefore $\cos EF = \sin \frac{1}{2}b \sin \frac{1}{2}c + \cos \frac{1}{2}b \cos \frac{1}{2}c \cos A$............(1)

If the sides of the triangle are small compared with the radius of the sphere, EF will not differ much from A; suppose $EF = A - \theta$, then approximately

$$\cos EF = \cos A + \theta \sin A;$$

and
$$\sin \tfrac{1}{2}b \sin \tfrac{1}{2}c = \sin^2 \tfrac{1}{4}(b+c) - \sin^2 \tfrac{1}{4}(b-c),$$
$$\cos \tfrac{1}{2}b \cos \tfrac{1}{2}c = \cos^2 \tfrac{1}{4}(b+c) - \sin^2 \tfrac{1}{4}(b-c);$$

therefore
$$\cos A + \theta \sin A = \sin^2 \tfrac{1}{4}(b+c) - \sin^2 \tfrac{1}{4}(b-c) + \{1 - \sin^2 \tfrac{1}{4}(b+c) - \sin^2 \tfrac{1}{4}(b-c)\} \cos A;$$

therefore
$$\theta \sin A = (1 - \cos A) \sin^2 \tfrac{1}{4}(b+c) - (1 + \cos A) \sin^2 \tfrac{1}{4}(b-c),$$

therefore $\quad \theta = \tan \frac{1}{2}A \sin^2 \frac{1}{4}(b+c) - \cot \frac{1}{2}A \sin^2 \frac{1}{4}(b-c)$. ....(2)

This gives the *circular measure* of θ; the number of seconds in the angle is found by dividing the circular measure by the circular measure of one second, or by multiplying by 206265, the number of seconds in a radian. If the lengths of the arcs corresponding to a and b respectively be α and β, and r the radius of the sphere, we have $\frac{\alpha}{r}$ and $\frac{\beta}{r}$ as the circular measures of a and b respectively; and the lengths of the sides of the chordal triangle are $2r \sin \frac{\alpha}{2r}$ and $2r \sin \frac{\beta}{2r}$ respectively. Thus when the sides of the spherical triangle and the radius of the sphere are known, we can calculate the angles and sides of the chordal triangle.

214. Legendre's theorem.* *If the sides of a spherical triangle are small compared with the radius of the sphere, then each angle*

* LEGENDRE, *Mémoires de Paris*, 1787, p. 338; *Trigonométrie*, Appendix V. Cf. GAUSS, *Disquisitiones generales circa superficies curvas*, §§ 27, 28, and MERTENS, *Schlömilch's Zeitschrift*, 1875.

of the spherical triangle exceeds by one third of the spherical excess the corresponding angle of the plane triangle whose sides are of the same lengths as the arcs of the spherical triangle.

Let A, B, C be the angles of the spherical triangle; a, b, c the sides; r the radius of the sphere; α, β, γ the lengths of the arcs which form the sides, so that $\frac{\alpha}{r}, \frac{\beta}{r}, \frac{\gamma}{r}$ are the circular measures of a, b, c respectively. Then

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c};$$

now

$$\cos a = 1 - \frac{\alpha^2}{2r^2} + \frac{\alpha^4}{24r^4} - \dots, \qquad \text{(3)}$$

$$\sin a = \frac{\alpha}{r} - \frac{\alpha^3}{6r^3} + \dots \qquad \text{(4)}$$

Similar expressions hold for $\cos b$ and $\sin b$, and for $\cos c$ and $\sin c$ respectively. Hence, if we neglect powers of the circular measure above the *fourth*, we have

$$\cos A = \frac{1 - \frac{\alpha^2}{2r^2} + \frac{\alpha^4}{24r^4} - \left(1 - \frac{\beta^2}{2r^2} + \frac{\beta^4}{24r^4}\right)\left(1 - \frac{\gamma^2}{2r^2} + \frac{\gamma^4}{24r^4}\right)}{\frac{\beta\gamma}{r^2}\left(1 - \frac{\beta^2}{6r^2}\right)\left(1 - \frac{\gamma^2}{6r^2}\right)}$$

$$= \frac{\frac{1}{2r^2}(\beta^2 + \gamma^2 - \alpha^2) + \frac{1}{24r^4}(\alpha^4 - \beta^4 - \gamma^4 - 6\beta^2\gamma^2)}{\frac{\beta\gamma}{r^2}\left(1 - \frac{\beta^2 + \gamma^2}{6r^2}\right)}$$

$$= \frac{1}{2\beta\gamma}\left\{\beta^2 + \gamma^2 - \alpha^2 + \frac{1}{12r^2}(\alpha^4 - \beta^4 - \gamma^4 - 6\beta^2\gamma^2)\right\}\left\{1 + \frac{\beta^2 + \gamma^2}{6r^2}\right\}$$

$$= \frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma} + \frac{\alpha^4 + \beta^4 + \gamma^4 - 2\beta^2\gamma^2 - 2\gamma^2\alpha^2 - 2\alpha^2\beta^2}{24\beta\gamma r^2} \qquad \text{(5)}$$

Now let A′, B′, C′ be the angles of the plane triangle whose sides are α, β, γ respectively; then

$$\left.\begin{aligned} \cos A' &= \frac{\beta^2+\gamma^2-\alpha^2}{2\beta\gamma}, \\ \text{and} \quad 2\beta^2\gamma^2+2\gamma^2\alpha^2+2\alpha^2\beta^2-\alpha^4-\beta^4-\gamma^4 &= 4\beta^2\gamma^2\sin^2 A', \end{aligned}\right\} \quad \text{...(6)}$$

thus
$$\cos A = \cos A' - \frac{\beta\gamma\sin^2 A'}{6r^2}. \quad \text{.....................(7)}$$

Suppose $A = A' + \theta$; then

$$\cos A = \cos A' - \theta \sin A' \text{ approximately};$$

therefore
$$\theta = \frac{\beta\gamma\sin A'}{6r^2} = \frac{S'}{3r^2}, \quad \text{.......................(8)}$$

where S′ denotes the area of the plane triangle whose sides are α, β, γ. Similarly

$$B = B' + \frac{S'}{3r^2}, \text{ and } C = C' + \frac{S'}{3r^2}; \quad \text{...............(9)}$$

hence approximately

$$A + B + C = A' + B' + C' + \frac{S'}{r^2} = \pi + \frac{S'}{r^2}; \quad \text{...........(10)}$$

therefore $\frac{S'}{r^2}$ is approximately equal to the spherical excess E of the spherical triangle, and thus it is established that

$$A = A' + \tfrac{1}{3}E, \quad B = B' + \tfrac{1}{3}E, \quad C = C' + \tfrac{1}{3}E.$$

215. It must be noticed that, if S denote the area of the spherical triangle, the spherical excess is *exactly* equal to $\frac{S}{r^2}$. Hence S and S′ are equal to one another, to the degree of approximation here employed. Thus the areas of the spherical triangle and of the plane triangle with sides of the same length differ from one another only by a small quantity of the order of the ratio that either area bears to the whole area of the sphere.

216. Approximate solution of triangles. LEGENDRE'S Theorem may be used for the approximate solution of spherical triangles in the following manner.

(1) *Suppose the three sides of a spherical triangle known*; then the values of α, β, γ are known, and by the formulae of Plane Trigonometry we can calculate S' and A', B', C'; then A, B, C are known from the formulae

$$A = A' + \frac{S'}{3r^2},\ B = B' + \frac{S'}{3r^2},\ C = C' + \frac{S'}{3r^2}. \quad \text{.........}(9)$$

(2) *Suppose two sides and the included angle of a spherical triangle known*, for example A, b, c. Then

$$S' = \tfrac{1}{2}\beta\gamma \sin A' = \tfrac{1}{2}\beta\gamma \sin A \text{ approximately.}$$

Then A' is known from the formula $A' = A - \frac{S'}{3r^2}$. Thus in the plane triangle two sides and the included angle are known; therefore its remaining parts can be calculated, and then those of the spherical triangle become known.

(3) *Suppose two sides and the angle opposite to one of them in a spherical triangle known*, for example A, a, b. Then

$$S' = \tfrac{1}{2}\alpha\beta \sin C' = \tfrac{1}{2}\alpha\beta \sin(A' + B')$$

$$= \tfrac{1}{2}\alpha\beta \sin(A + B') \text{ approximately,}$$

and so

$$E = \frac{\alpha\beta}{2r^2} \sin(A + B'). \quad \text{.....................}(11)$$

The angle B' is not given; but, for the purpose of substitution in the expression for E, its values are obtained to a sufficient approximation from the relation

$$\sin B' = \frac{\beta}{\alpha} \sin A' = \frac{\beta}{\alpha} \sin A.$$

When E has been obtained thus, A' is got from the formula $A = A' + \frac{1}{3}E$, and the problem is reduced to the solution of the plane triangle whose elements α, β, and A' are known.

(4) *Suppose two angles and the included side of a spherical triangle known*, for example A, B, c.

Then $$S' = \frac{\gamma^2 \sin A' \sin B'}{2\sin(A'+B')} = \frac{\gamma^2 \sin A \sin B}{2\sin(A+B)} \text{ nearly.} \quad \text{..........(12)}$$

This gives the value of E, and so enables us to deduce the values of A′ and B′ from those of A and B. Hence in the plane triangle two angles and the included side are known.

(5) *Suppose two angles and the side opposite to one of them in a spherical triangle known*, for example A, B, a. Then

$$C' = \pi - A' - B' = \pi - A - B \text{ approximately,}$$

and $$S' = \frac{a^2 \sin B' \sin C'}{2\sin A'},$$

$$= \tfrac{1}{2}a^2 \sin B \sin(A+B) \operatorname{cosec} A, \text{ approximately.} \quad \text{...(13)}$$

Thus E is obtained, and the corrected values of A′ and B′ deduced from A and B in the usual manner. It only remains to solve the plane triangle whose elements A′, B′, and a are known. This case is free from ambiguity, because we are dealing only with triangles whose sides are small compared with r.

217. The importance of LEGENDRE'S Theorem in the application of Spherical Trigonometry to the measurement of the Earth's surface has given rise to various developments of it which enable us to test the degree of exactness of the approximation. We shall consider some of these developments. We have seen that the spherical excess is approximately equal to $\frac{S'}{r^2}$, and we shall begin by investigating a closer approximate formula for the spherical excess.

218. To find an approximate value of the spherical excess.*
By L'HUILIER'S theorem, Art. 134,

$$\tan \tfrac{1}{4}E = \left[\tan \tfrac{1}{2}s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c)\right]^{\frac{1}{2}}, \quad \text{...(14)}$$

* GAUSS, *Disquisitiones, etc.*, § 29.

and therefore, approximately,

$$=\left[\frac{1}{16}s\left\{1+\frac{s^2}{12}\right\}(s-a)\left\{1+\frac{(s-a)^2}{12}\right\}\right.$$

$$\left.(s-b)\left\{1+\frac{(s-b)^2}{12}\right\}(s-c)\left\{1+\frac{(s-c)^2}{12}\right\}\right]^{\frac{1}{2}},$$

$$=\tfrac{1}{4}[s(s-a)(s-b)(s-c)]^{\frac{1}{2}}\left[1+\frac{s^2+(s-a)^2+(s-b)^2+(s-c)^2}{12}\right]^{\frac{1}{2}} \quad (15)$$

$$=\frac{\mathsf{S}'}{4r^2}\left\{1+\frac{a^2+b^2+c^2}{24}\right\};$$

thus
$$\mathsf{E}=\frac{\mathsf{S}'}{r^2}\left(1+\frac{\alpha^2+\beta^2+\gamma^2}{24r^2}\right). \quad \text{.................}(16)$$

Hence, to this order of approximation, the area of the spherical triangle exceeds that of the plane triangle by the fraction $(\alpha^2+\beta^2+\gamma^2)/24r^2$ of the latter.

Another expression, which is sometimes used, is derived from formula (5) of Art. 132,

$$\sin\tfrac{1}{2}\mathsf{E}=\sin\mathsf{C}\sin\tfrac{1}{2}a\sin\tfrac{1}{2}b\sec\tfrac{1}{2}c. \quad \text{.............}(17)$$

For, from this, approximately

$$\sin\tfrac{1}{2}\mathsf{E}=\sin\mathsf{C}\,\frac{\alpha\beta}{4r^2}\left(1-\frac{\alpha^2}{24r^2}\right)\left(1-\frac{\beta^2}{24r^2}\right)\left(1-\frac{\gamma^2}{8r^2}\right)^{-1}$$

and therefore

$$\mathsf{E}=\sin\mathsf{C}\,\frac{\alpha\beta}{2r^2}\left(1+\frac{3\gamma^2-\alpha^2-\beta^2}{24r^2}\right). \quad \text{..................}(18)$$

Of course (16) may be derived from (18) by substituting $\sin\mathsf{C}'+\tfrac{1}{3}\mathsf{E}\cos\mathsf{C}'$ for $\sin\mathsf{C}$.

219. To find a closer approximation to the value of A.*

Since $\sin\tfrac{1}{2}\mathsf{A}\cos\tfrac{1}{2}\mathsf{A}'=\sqrt{\dfrac{\sin(s-b)\sin(s-c)}{\sin b\sin c}\cdot\dfrac{s(s-a)}{bc}},$

and $\cos\tfrac{1}{2}\mathsf{A}\sin\tfrac{1}{2}\mathsf{A}'=\sqrt{\dfrac{\sin s\sin(s-a)}{\sin b\sin c}\cdot\dfrac{(s-b)(s-c)}{bc}},$

* GAUSS, *Disquisitiones, etc.*, § 27 ; CLARKE'S *Geodesy*, p. 47.

therefore $\sin\frac{1}{2}(A - A')$

$$=\frac{S'}{r^2bc}\cdot\frac{\left\{\frac{\sin(s-b)}{s-b}\cdot\frac{\sin(s-c)}{s-c}\right\}^{\frac{1}{2}}-\left\{\frac{\sin s}{s}\cdot\frac{\sin(s-a)}{s-a}\right\}^{\frac{1}{2}}}{\left\{\frac{\sin b}{b}\cdot\frac{\sin c}{c}\right\}^{\frac{1}{2}}} \quad\ldots(19)$$

If we substitute for the square-roots of the sines expressions given by the approximate formula

$$\sin^{\frac{1}{2}}\theta=\theta^{\frac{1}{2}}\left(1-\tfrac{1}{12}\theta^2+\tfrac{1}{1440}\theta^4\right), \quad\ldots\ldots(20)$$

and notice that

$$s^2+(s-a)^2-(s-b)^2-(s-c)^2=2bc, \quad\ldots\ldots(21)$$

$$s^4+(s-a)^4-(s-b)^4-(s-c)^4=bc(3a^2+b^2+c^2), \quad\ldots\ldots(22)$$

$$(s-b)^2(s-c)^2-s^2(s-a)^2=\tfrac{1}{2}bc(a^2-b^2-c^2), \quad\ldots\ldots(23)$$

we readily get

$$A-A'=\frac{1}{3}\frac{S'}{r^2}\left(1+\frac{\alpha^2+7\beta^2+7\gamma^2}{120r^2}\right). \quad\ldots\ldots(24)$$

Now replacing S' by its value in terms of E, as given by result (16) of the previous Article, we finally get

$$A=A'+\tfrac{1}{3}E+\frac{1}{180}\frac{E}{r^2}\left(-2\alpha^2+\beta^2+\gamma^2\right). \quad\ldots\ldots(25)$$

220. To find an approximate value of $\frac{\sin A}{\sin B}$.

$$\frac{\operatorname{Sin} A}{\sin B}=\frac{\sin a}{\sin b};$$

hence approximately

$$\frac{\sin A}{\sin B}=\frac{\alpha\left(1-\frac{\alpha^2}{6r^2}+\frac{\alpha^4}{120r^4}\right)}{\beta\left(1-\frac{\beta^2}{6r^2}+\frac{\beta^4}{120r^4}\right)}$$

$$=\frac{\alpha}{\beta}\left(1-\frac{\alpha^2}{6r^2}+\frac{\alpha^4}{120r^4}+\frac{\beta^2}{6r^2}-\frac{\alpha^2\beta^2}{36r^4}-\frac{\beta^4}{120r^4}+\frac{\beta^4}{36r^4}\right)$$

$$=\frac{\alpha}{\beta}\left\{1+\frac{\beta^2-\alpha^2}{6r^2}\left(1+\frac{7\beta^2-3\alpha^2}{60r^2}\right)\right\}. \quad\ldots\ldots(26)$$

221. To express $\cot B - \cot A$ **approximately.**

$$\text{Cot } B - \cot A = \frac{1}{\sin B}\left(\cos B - \frac{\sin B}{\sin A}\cos A\right);$$

hence, approximately, by Art. 220,

$$\cot B - \cot A = \frac{1}{\sin B}\left(\cos B - \frac{\beta}{\alpha}\cos A - \frac{\beta}{\alpha}\,\frac{\alpha^2 - \beta^2}{6r^2}\cos A\right).$$

Now we have shewn, in Art. 214, that approximately

$$\cos A = \frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma} + \frac{\alpha^4 + \beta^4 + \gamma^4 - 2\beta^2\gamma^2 - 2\gamma^2\alpha^2 - 2\alpha^2\beta^2}{24\beta\gamma r^2},$$

and $\cos B$ is equal to an expression of the same type;

therefore $$\cos B - \frac{\beta}{\alpha}\cos A = \frac{\alpha^2 - \beta^2}{\alpha\gamma} \text{ approximately.}$$

Thus $$\cot B - \cot A = \frac{\alpha^2 - \beta^2}{\alpha\gamma \sin B} - \frac{\alpha^2 - \beta^2}{\alpha\gamma \sin B}\,\frac{\beta^2 + \gamma^2 - \alpha^2}{12r^2}$$

$$= \frac{\alpha^2 - \beta^2}{\alpha\gamma \sin B}\left(1 - \frac{\beta^2 + \gamma^2 - \alpha^2}{12r^2}\right). \quad\ldots\ldots\ldots (27)$$

222. LEGENDRE'S theorem is extensively used in practical geodesy, where it greatly simplifies the solution of triangles. It is desirable, however, to enquire how far it can be used with safety.

To find an approximate value of the error in the length of a side of a spherical triangle when calculated by Legendre's theorem.

Suppose the side β known and the side α required; if E be the spherical excess of the triangle, we have, by LEGENDRE'S theorem,

$$\alpha = \beta\,\frac{\sin(A - \frac{1}{3}E)}{\sin(B - \frac{1}{3}E)}.$$

Let $\delta\alpha$ be the error of the side so computed. Its value will depend on the value of E actually adopted, which may be calculated in more than one way. We shall simplify the

expression for the error as far as possible before substituting for E.

$$\delta a = \beta \frac{\sin (A - \frac{1}{3}E)}{\sin (B - \frac{1}{3}E)} - a \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

Now approximately

$$\frac{\sin (A - \frac{1}{3}E)}{\sin (B - \frac{1}{3}E)} = \frac{\sin A - \frac{1}{3}E \cos A - \frac{1}{18}E^2 \sin A}{\sin B - \frac{1}{3}E \cos B - \frac{1}{18}E^2 \sin B}$$

$$= \frac{\sin A}{\sin B}\left(1 - \tfrac{1}{3}E \cot A - \tfrac{1}{18}E^2\right)\left(1 - \tfrac{1}{3}E \cot B - \tfrac{1}{18}E^2\right)^{-1}$$

$$= \frac{\sin A}{\sin B}\left\{1 + \tfrac{1}{3}E(\cot B - \cot A) + \tfrac{1}{9}E^2 \cot B(\cot B - \cot A)\right\}$$

$$= \frac{\sin A}{\sin B} + \tfrac{1}{3}E\frac{\sin A}{\sin B}\left(\cot B - \cot A\right)\left(1 + \tfrac{1}{3}E \cot B\right). \ldots\ldots\ldots (29)$$

Also the following formulae are true so far as terms involving r^2:

$$\frac{\sin A}{\sin B} = \frac{a}{\beta}\left(1 + \frac{\beta^2 - a^2}{6r^2}\right), \ldots\ldots\ldots\ldots\ldots\ldots (30)$$

$$\cot B - \cot A = \frac{a^2 - \beta^2}{a\gamma \sin B}\left(1 - \frac{\beta^2 + \gamma^2 - a^2}{12r^2}\right). \ldots\ldots\ldots (31)$$

These may be substituted in the term of first order in E; while the *first* approximation to E, namely $\frac{1}{2}\frac{a\gamma}{r^2}\sin B$, is sufficient for substitution in the terms of order E^2. In the term which is not small we must substitute the value of $\frac{\sin A}{\sin B}$ obtained in Art. 220.

Thus we get

$$\frac{\sin (A - \frac{1}{3}E)}{\sin (B - \frac{1}{3}E)} = \frac{a}{\beta}\left\{1 + \frac{\beta^2 - a^2}{6r^2}\left(1 + \frac{7\beta^2 - 3a^2}{60r^2}\right)\right\}$$
$$+ \frac{1}{3}E\frac{a}{\beta}\left(1 + \frac{\beta^2 - a^2}{6r^2}\right)\frac{a^2 - \beta^2}{a\gamma \sin B}\left(1 - \frac{\beta^2 + \gamma^2 - a^2}{12r^2}\right)$$
$$+ \frac{1}{36}\,\frac{a^2\gamma^2 \sin^2 B}{r^4}\,\frac{a^2 - \beta^2}{\beta\gamma \sin B}\cot B; \ldots\ldots (32)$$

multiplying by β, subtracting a, and again introducing the first approximation to E in the terms of order E^2, we get after some straightforward reduction

$$\delta a = \frac{a(\beta^2 - a^2)}{6}\left[-\frac{2E}{a\gamma \sin B} + \frac{1}{r^2} + \frac{7\beta^2 - 3a^2}{60r^4}\right]. \qquad \text{.......(33)}$$

223. If, therefore, we calculate E from the formula

$$E = \frac{a\gamma \sin B}{2r^2},$$

the error is

$$\delta a = \frac{a(\beta^2 - a^2)(7\beta^2 - 3a^2)}{360r^4}. \qquad \text{................(34)}$$

But if we compute E from the equation corresponding to (18) of Art. 218, we have

$$E = \frac{a\gamma \sin B}{2r^2}\left(1 + \frac{3\beta^2 - a^2 - \gamma^2}{24r^2}\right), \qquad \text{..............(35)}$$

and therefore the error of a is

$$\delta a = \frac{a(\beta^2 - a^2)(5\gamma^2 - a^2 - \beta^2)}{720r^4}. \qquad \text{.................(36)}$$

The errors of γ in the two cases are represented by corresponding expressions.

224. To take a numerical example,* suppose the sides of a triangle on the surface of the earth to be $a = 220$, $\beta = 60$, $\gamma = 180$ miles. If we use the first method of calculating the spherical excess the errors of the resulting sides are

$$\delta a = +0{\cdot}068, \quad \delta\gamma = +0{\cdot}026, \quad \text{in feet.}$$

If we use the second method the errors are

$$\delta a = -0{\cdot}031, \quad \delta\gamma = -0{\cdot}030, \quad \text{in feet.}$$

Thus it is seen that the errors resulting from the use of LEGENDRE'S theorem are very minute.

225. Approximate solution of a triangle having one side small. When only one of the sides of the triangle is small

* CLARKE'S *Geodesy*, p. 49.

compared with the radius of the sphere, the following approximations are useful; they are given by Col. CLARKE in his work on *Geodesy*.

To obtain an approximate solution of a triangle right-angled at C, *when* a *and* b *are given and* b *is very small.*

If V denote the complement of A, it is readily seen that V and B are in the present instance very small.

Now $$\frac{\tan B}{\tan b} = \frac{1}{\sin a},$$

and if, for brevity, we denote the right side by k, and expand the tangents in series, we get

$$B + \tfrac{1}{3}B^3 + \tfrac{2}{15}B^5 + \ldots = k(b + \tfrac{1}{3}b^3 + \tfrac{2}{15}b^5 + \ldots), \quad \ldots\ldots(37)$$

whence it can be inferred that

$$\frac{B}{b} = k\{1 + \tfrac{1}{3}b^2(1 - k^2) + \tfrac{1}{15}b^4(1 - k^2)(2 - 3k^2)\ldots\}. \quad \ldots\ldots(38)$$

Replacing the value of k, and bearing in mind that

$$\sin(a + x) = \sin a(1 + x \cot a),$$

when the square of x is neglected, we find

$$B = \frac{b}{\sin(a + \tfrac{1}{3}b^2 \cot a)}, \quad \ldots\ldots(39)$$

the terms omitted in the approximation being of order b^5.

Again, $\sin V = \sin B \cos a$,

therefore $$V = B \cos a(1 - \tfrac{1}{6}b^2 - \ldots), \quad \ldots\ldots(40)$$

which, when terms in b^5 are neglected, may be put in the form

$$V = B \cos(a + \tfrac{1}{6}b^2 \cot a). \quad \ldots\ldots(41)$$

Further, if c exceed a by the small quantity x,

$$\cos(a + x) = \cos a \cos b,$$

therefore $$x = \tfrac{1}{2}b^2 \cot a - \tfrac{1}{24}b^4(1 + 3\cot^2 a)\cot a. \quad \ldots\ldots(42)$$

Thus, supposing that we may omit the fourth power of b, the solution of the triangle is:

$$\left.\begin{aligned} c - a &= \tfrac{1}{2}b^2 \cot a = \eta, \\ \mathsf{B} &= \frac{b}{\sin(a + \frac{2}{3}\eta)}, \\ 90° - \mathsf{A} &= \mathsf{B}\cos(a + \tfrac{1}{3}\eta). \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(43)$$

226. *To obtain an approximate solution of the oblique-angled triangle when* A, b, *and* C *are given, and* b *is very small.*

In the more general case in which b is small, but C not a right angle, the series already employed may be used to find $c - a$.

For
$$\frac{\tan\frac{1}{2}(c-a)}{\tan\frac{1}{2}b} = \frac{\sin\frac{1}{2}(\mathsf{C}-\mathsf{A})}{\sin\frac{1}{2}(\mathsf{C}+\mathsf{A})},$$

and, if we denote the right-hand member by k, this leads to

$$\frac{c-a}{b} = k\{1 + \tfrac{1}{12}b^2(1-k^2) + \tfrac{1}{240}b^4(1-k^2)(2-3k^2)\ldots\}. \;..(44)$$

If $c - a$ be computed by this formula, b being small, and k numerically less than unity, the error involved in omitting the term b^5 is very small; for the greatest numerical value that $k(1-k^2)(2-3k^2)$ can assume is 0·51. If then b be as much as 2° the term in b^5 amounts at a maximum to 0″·000022; it may therefore in all cases be neglected, and so we have

$$c - a = b\frac{\sin\frac{1}{2}(\mathsf{C}-\mathsf{A})}{\sin\frac{1}{2}(\mathsf{C}+\mathsf{A})}\left\{1 + \frac{b^2}{12}\,\frac{\sin\mathsf{C}\sin\mathsf{A}}{\sin^2\frac{1}{2}(\mathsf{C}+\mathsf{A})}\right\}. \;\ldots\ldots(45)$$

The following results, for a triangle in which b is so small that b^3 may be neglected, are due to Mr. T. J. I'A. Bromwich:

$$\left.\begin{aligned} c - a &= b\cos\mathsf{A} - \tfrac{1}{2}b^2\cot c\sin^2\mathsf{A}, \\ \mathsf{B}\sin c &= b\sin\mathsf{A} + \tfrac{1}{2}b^2\cot c\sin 2\mathsf{A}, \\ \pi - \mathsf{C} - \mathsf{A} &= b\sin\mathsf{A}\cot c + \tfrac{1}{4}b^2\sin 2\mathsf{A}(1 + 2\cot^2 c). \end{aligned}\right\} \ldots\ldots\ldots(46)$$

They afford an approximate solution when b, A, and c are given.

EXAMPLES XIV.

MISCELLANEOUS EXAMPLES.

1. If the sides of a spherical triangle AB, AC be produced to B′, C′, so that BB′, CC′ are the semi-supplements of AB, AC respectively, shew that the arc B′C′ will subtend an angle at the centre of the sphere equal to the angle between the chords of AB and AC.

2. Deduce LEGENDRE's theorem from the formula

$$\tan^2\frac{A}{2}=\frac{\sin\frac{1}{2}(a+b-c)\sin\frac{1}{2}(c+a-b)}{\sin\frac{1}{2}(b+c-a)\sin\frac{1}{2}(a+b+c)}.$$

3. If a quadrilateral ABCD be inscribed in a small circle on a sphere so that two opposite angles A and C may be at opposite extremities of a diameter, the sum of the cosines of the sides is constant.

4. In a spherical triangle if $A=B=2C$, shew that

$$\cos a\cos\tfrac{1}{2}a=\cos(c+\tfrac{1}{2}a).$$

5. ABC is a spherical triangle each of whose sides is a quadrant; P is any point within the triangle; shew that

$$\cos PA\cos PB\cos PC+\cot BPC\cot CPA\cot APB=0,$$

and

$$\tan ABP\tan BCP\tan CAP=1.$$

6. If O be the middle point of an equilateral triangle ABC, and P any point on the surface of the sphere, then

$$\tfrac{1}{4}(\tan PO\tan OA)^2(\cos PA+\cos PB+\cos PC)^2$$
$$=\cos^2 PA+\cos^2 PB+\cos^2 PC-\cos PB\cos PC-\cos PC\cos PA-\cos PA\cos PB.$$

7. If ABC be a triangle having each side a quadrant, O the pole of the inscribed circle, P any point on the sphere, then

$$(\cos PA+\cos PB+\cos PC)^2=3\cos^2 PO.$$

8. From each of three points on the surface of a sphere arcs are drawn on the surface to three other points situated on a great circle of the sphere, and their cosines are a, b, c; a', b', c'; a'', b'', c''. Shew that $ab''c'+a'bc''+a''b'c=ab'c''+a'b''c+a''bc'$.

9. From Arts 220 and 221, shew that approximately

$$\log\beta=\log a+\log\sin B-\log\sin A+\frac{S'}{3r^2}(\cot A-\cot B).$$

10. By continuing the approximation in Art. 214 so as to include the terms involving r^4, shew that approximately

$$\cos A = \cos A' - \frac{\beta\gamma \sin^2 A'}{6r^2} + \frac{\beta\gamma(\alpha^2 - 3\beta^2 - 3\gamma^2)\sin^2 A'}{180r^4}.$$

11. From the preceding result shew that if $A = A' + \theta$ then approximately

$$\theta = \frac{\beta\gamma \sin A'}{6r^2}\left(1 + \frac{\alpha^2 + 7\beta^2 + 7\gamma^2}{120r^2}\right).$$

12. An equilateral triangle is described on a sphere whose radius is r, and each of its sides is less than a third of the circumference of a great circle by a small difference k. Shew that the square of a side of the polar triangle is $4rk\sqrt{3}$ very nearly.

CHAPTER XIII.

GEODETICAL OPERATIONS.

227. One of the most important applications of Trigonometry, both Plane and Spherical, is to the determination of the figure and dimensions of the Earth itself, and of any portion of its surface. We shall give a brief outline of the subject, and for further information refer to the article *Geodesy* in the *Encyclopaedia Britannica*, to AIRY'S treatise on the *Figure of the Earth* in the *Encyclopaedia Metropolitana*, and to Colonel A. R. CLARKE'S work on *Geodesy*, published by the Clarendon Press, 1880. For practical knowledge of the details of the operations it will be necessary to study some of the published accounts of the great surveys which have been effected in different parts of the world, as for example, the *Account of the measurement of two sections of the Meridional arc of India*, by Lieut.-Colonel EVEREST, 1847; or the *Account of the Observations and Calculations of the Principal Triangulation in the Ordnance Survey of Great Britain and Ireland*, 1858.

The Ordnance Survey, by Lieut.-Colonel T. P. WHITE, 1886, gives a popular account of the national survey, without technical details.

228. An important part of any survey consists in the measurement of a horizontal line, which is called a *base*. A level plain of a few miles in length is selected and a line is measured on it with every precaution to ensure accuracy. Rods of deal, and of metal, hollow tubes of glass, and steel chains, have been used in different surveys; the temperature

is carefully observed during the operations, and allowance is made for the varying lengths of the rods or chains, which arise from variations in the temperature.

229. At various points of the country suitable stations are selected and signals erected; then, by supposing lines to be drawn connecting the signals, the country is divided into a series of triangles. The angles of these triangles are observed, that is, the angles which any two signals subtend at a third. For example suppose A and B to denote the extremities of the *base*, and C a signal at a third point visible from A and B; then in the triangle ABC the angles ABC and BAC are observed, and then AC and BC can be calculated. Again, let D be a signal at a fourth point, such that it is visible from C and A; then the angles ACD and CAD are observed, and as AC is known, CD and AD can be calculated.

230. Besides the original *base* other lines are measured in convenient parts of the country surveyed, and their measured lengths are compared with their lengths obtained by calculation through a series of triangles from the original base. The degree of closeness with which the measured length agrees with the calculated length is a test of the accuracy of the survey. During the progress of the Ordnance Survey of Great Britain and Ireland several lines have been measured; the last two are, one near Lough Foyle in Ireland, which was measured in 1827 and 1828, and one on Salisbury Plain, which was measured in 1849. The line near Lough Foyle is nearly 8 miles long, and the line on Salisbury Plain is nearly 7 miles long; and the difference between the length of the line on Salisbury Plain as measured and as calculated from the Lough Foyle base is less than 5 inches (*An Account of the Observations* ... page 419).

231. There are different methods of effecting the calculations for determining the lengths of the sides of all the triangles in

the survey. One method is to use the exact formulae of Spherical Trigonometry. The radius of the Earth may be considered known very approximately, let this radius be denoted by r; then, if a be the length of any arc, the circular measure of the angle which the arc subtends at the centre of the earth is a/r. The formulae of Spherical Trigonometry give expressions for the trigonometrical functions of a/r, so that a/r may be found and then a. Since in practice a/r is always very small, it becomes necessary to pay attention to the methods of securing accuracy in calculations which involve the logarithmic trigonometrical functions of small angles, (*Plane Trigonometry*, Art. 205).

Instead of the exact calculation of the triangles by Spherical Trigonometry, various methods of approximation have been proposed; only two of these methods however have been much used. One method of approximation consists in deducing from the angles of the spherical triangles the angles of the *chordal triangles*, and then computing the latter triangles by Plane Trigonometry (see Art. 213). The other method of approximation consists in the use of LEGENDRE'S Theorem, (see Art. 214).

232. The three methods which we have indicated were all used by DELAMBRE in calculating the triangles in the French survey (*Base du Système Métrique*, Tome III, page 7). In the earlier operations of the trigonometrical survey of Great Britain and Ireland, the triangles were calculated by the chord method; but this has been for many years discontinued, and in place of it LEGENDRE'S Theorem has been universally adopted (*An Account of the Observations* ... page 244). The triangles in the Indian Survey are stated by Lieut.-Colonel EVEREST to be computed on LEGENDRE'S Theorem. (*An Account of the Measurement* ... page CLVIII.)

233. If the three angles of a plane triangle be observed, the

fact that their sum ought to be equal to two right angles affords a test of the accuracy with which the observations are made. We shall proceed to shew how a test of the accuracy of observations of the angles of a spherical triangle formed on the Earth's surface may be obtained by means of the *spherical excess.*

234. *The area of a spherical triangle formed on the Earth's surface being known in square feet, it is required to establish a rule for computing the spherical excess in seconds.*

Let n be the number of seconds in the spherical excess, s the number of square feet in the area of the triangle, r the number of feet in the radius of the Earth. Then if E be the circular measure of the spherical excess,

$$s = \mathsf{E}r^2,$$

and
$$\mathsf{E} = \frac{n\pi}{180 \,.\, 60 \,.\, 60} = \frac{n}{206265} \text{ approximately};$$

therefore
$$s = \frac{nr^2}{206265}.$$

Now by actual measurement the mean length of a degree on the Earth's surface is found to be 365155 feet; thus

$$\frac{\pi r}{180} = 365155.$$

With the value of r obtained from this equation it is found, by logarithmic calculation, that

$$\log n = \log s - 9{\cdot}326774.$$

Hence n is known when s is known.

This formula is called General ROY'S rule, as it was used by him in the trigonometrical survey of Great Britain and Ireland. Mr. DAVIES, however, claims it for Mr. DALBY. (See HUTTON'S *Course of Mathematics*, by DAVIES, Vol. II. p. 47.)

235. In order to apply General ROY'S rule, we must know

the area of the spherical triangle. Now the area is not known *exactly* unless the elements of the spherical triangle are known *exactly*; but it is found that in such cases as occur in practice an approximate value of the area is sufficient. Suppose, for example, that we use the area of the *plane triangle* considered in LEGENDRE'S Theorem, instead of the area of the *spherical triangle itself*; then it appears from Art. 218, that the error is approximately denoted by the fraction $\frac{\alpha^2+\beta^2+\gamma^2}{24r^2}$ of the former area, and this fraction is less than ·0001, if the sides do not exceed 100 miles in length. Or again, suppose we want to estimate the influence of errors in the angles on the calculation of the area; let the circular measure of an error be h, so that instead of $\frac{1}{2}\alpha\beta \sin \mathrm{C}$ we ought to use $\frac{1}{2}\alpha\beta \sin(\mathrm{C}+h)$; the error then bears to the area approximately the ratio expressed by $h \cot \mathrm{C}$. Now in modern observations h will not exceed the circular measure of a few seconds, so that, if C be not very small, $h \cot \mathrm{C}$ is practically insensible.

236. In geodetical operations the largest triangles practicable for observations are so small compared with the earth's surface that their spherical excess does not as a rule amount to many seconds. It requires a triangle containing about 76 square miles to produce one second of excess. In a few of the greatest triangles of the English survey the excess is more than 30 seconds; the maximum reached was 64 seconds.

237. The following example was selected by WOODHOUSE from the triangles of the English survey, and has been adopted by other writers. The observed angles of a triangle being respectively 42° 2′ 32″, 67° 55′ 39″, 70° 1′ 48″, the sum of the errors made in the observations is required, supposing the side opposite to the angle A to be 27404·2 feet. The area is calculated from the expression $\frac{a^2 \sin \mathrm{B} \sin \mathrm{C}}{2 \sin \mathrm{A}}$, and by General

ROY'S rule it is found that $n = \cdot 23$. Now the sum of the observed angles is $180° - 1''$, and as it ought to have been $180° + 0''\cdot 23$, it follows that the sum of the errors of the observations is $1''\cdot 23$. This total error may be distributed among the observed angles in such proportion as the opinion of the observer may suggest; one way is to increase each of the observed angles by one-third of $1''\cdot 23$, and take the angles thus corrected for the true angles.

238. An investigation has been made with respect to the form of a triangle, in which errors in the observations of the angles will exercise the least influence on the lengths of the sides, and although the reasoning is allowed to be vague it may be deserving of the attention of the student. Suppose the three angles of a triangle observed, and one side, as a, known, it is required to find the form of the triangle in order that the other sides may be least affected by errors in the observations. The spherical excess of the triangle may be supposed known with sufficient accuracy for practice, and if the sum of the observed angles does not exceed two right angles by the proper spherical excess, let these angles be altered by adding the same quantity to each, so as to make their sum correct. Let A, B, C be the angles thus furnished by observation and altered if necessary; and let δA, δB, and δC denote the respective errors of A, B, and C. Then $\delta A + \delta B + \delta C = 0$, because by supposition the sum of A, B, and C is correct. Considering the triangle as approximately plane, the true value of the side c is

$$\frac{a \sin (C + \delta C)}{\sin (A + \delta A)}, \text{ that is, } \frac{a \sin (C + \delta C)}{\sin (A - \delta B - \delta C)}.$$

Now approximately

$$\sin (C + \delta C) = \sin C + \delta C \cos C,$$

$$\sin (A - \delta B - \delta C) = \sin A - (\delta B + \delta C) \cos A.$$

Hence approximately

$$c = \frac{a \sin C}{\sin A}\{1 + \delta C \cot C\}\{1 - (\delta B + \delta C)\cot A\}^{-1}$$

$$= \frac{a \sin C}{\sin A}\{1 + \delta B \cot A + \delta C(\cot C + \cot A)\};$$

and $\cot C + \cot A = \frac{\sin(A + C)}{\sin A \sin C} = \frac{\sin B}{\sin A \sin C}$ approximately.

Hence the error of c is approximately

$$\frac{a \sin B}{\sin^2 A}\delta C + \frac{a \sin C \cos A}{\sin^2 A}\delta B.$$

Similarly the error of b is approximately

$$\frac{a \sin C}{\sin^2 A}\delta B + \frac{a \sin B \cos A}{\sin^2 A}\delta C.$$

Now it is impossible to assign exactly the signs and magnitudes of the errors δB and δC, so that the reasoning must be vague. It is obvious that, to make the error small, $\sin A$ must not be small. And as the sum of δA, δB, and δC is zero, two of them must have the same sign, and the third the opposite sign; we may therefore consider that it is more probable that any two, as δB and δC, have different signs, than that they have the same sign.

If δB and δC have different signs the errors of b and c will be less when $\cos A$ is positive than when $\cos A$ is negative; A therefore ought to be less than a right angle. And if δB and δC are probably not very different, B and C should be nearly equal. These conditions will be satisfied by a triangle differing not much from an equilateral triangle.

If two angles only, A and B, be observed, we obtain the same expressions as before for the errors in b and c; but we have no reason for considering that δB and δC are of different signs rather than of the same sign. In this case, then, the supposition that A is a right angle will probably make the errors smallest.

239. The preceding article is taken from the Treatise on Trigonometry in the *Encyclopaedia Metropolitana*. The least satisfactory part is that in which it is considered that δB and δC may be supposed nearly equal; for since $\delta A + \delta B + \delta C = 0$, if we suppose δB and δC nearly equal and of opposite signs, we do in effect suppose $\delta A = 0$ nearly; thus in observing three angles, we suppose that in one observation a certain error is made, in a second observation the same numerical error is made but with an opposite sign, and in the remaining observation no error is made.

240. We have hitherto proceeded on the supposition that the Earth is a sphere; it is however approximately a spheroid of small eccentricity. For the small corrections which must in consequence be introduced into the calculations we must refer to the works named in Art. 227. One of the results obtained is that the error caused by regarding the Earth as a sphere instead of a spheroid increases with the departure of the triangle from the well-conditioned or equilateral form (*An Account of the Observations* ... page 243). Under certain circumstances the spherical excess is the same on a spheroid as on a sphere (*Figure of the Earth* in the *Encyclopaedia Metropolitana*, pages 198 and 215).

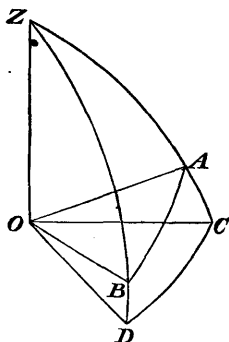
241. In geodetical operations it is sometimes required to determine the horizontal angle between two points, which are at a small angular distance from the horizon, the angle which the objects subtend being known, and also the angles of elevation or depression.

Suppose OA and OB the directions in which the two points are seen from O; and let the angle AOB be observed. Let OZ be the direction at right angles to the observer's horizon; describe a sphere round O as a centre, and let vertical planes through OA and OB meet the horizon at OC and OD respectively: then the angle COD is required.

Let AOB $=\theta$, COD $=\theta+x$, AOC $=h$, BOD $=k$; from the triangle AZB

$$\cos \text{AZB} = \frac{\cos\theta - \cos \text{ZA} \cos \text{ZB}}{\sin \text{ZA} \sin \text{ZB}} = \frac{\cos\theta - \sin h \sin k}{\cos h \cos k};$$

and $\cos \text{AZB} = \cos \text{COD} = \cos(\theta + x)$; thus

$$\cos(\theta+x) = \frac{\cos\theta - \sin h \sin k}{\cos h \cos k}$$

This formula is exact; by approximation we obtain

$$\cos\theta - x\sin\theta = \frac{\cos\theta - hk}{1 - \frac{1}{2}(h^2+k^2)};$$

therefore $\quad x\sin\theta = hk - \frac{1}{2}(h^2+k^2)\cos\theta$, nearly,

and
$$x = \frac{2hk - (h^2+k^2)(\cos^2\frac{1}{2}\theta - \sin^2\frac{1}{2}\theta)}{2\sin\theta}$$
$$= \tfrac{1}{4}(h+k)^2 \tan\tfrac{1}{2}\theta - \tfrac{1}{4}(h-k)^2 \cot\tfrac{1}{2}\theta.$$

This process, by which we find the angle COD from the angle AOB, is called *reducing an angle to the horizon.*

CHAPTER XIV.

ON SMALL VARIATIONS IN THE PARTS OF A SPHERICAL TRIANGLE.

242. It is sometimes important to know what amount of error will be introduced into one of the calculated parts of a triangle by reason of any small error which may exist in the given parts. We shall here consider an example.

243. A side and the opposite angle of a spherical triangle remain constant: determine the connexion between the small variations of any other pair of elements.

Suppose C and c to remain constant.

(i) Required the connexion between the small variations of the other sides. We suppose a and b to denote the sides of one triangle which can be formed with C and c as fixed elements, and $a+\delta a$ and $b+\delta b$ to denote the sides of another such triangle; then we require the ratio of δa to δb when both are extremely small. We have

$$\cos c = \cos a \cos b + \sin a \sin b \cos C,$$

and $$\cos c = \cos(a+\delta a)\cos(b+\delta b) + \sin(a+\delta a)\sin(b+\delta b)\cos C;$$

also $$\cos(a+\delta a) = \cos a - \sin a\,\delta a, \text{ nearly},$$

and $$\sin(a+\delta a) = \sin a + \cos a\,\delta a, \text{ nearly},$$

with similar formulae for $\cos(b+\delta b)$ and $\sin(b+\delta b)$. (See *Plane Trigonometry*, Chap. XII.) Thus

$$\cos c = (\cos a - \sin a\,\delta a)(\cos b - \sin b\,\delta b) \\ + (\sin a + \cos a\,\delta a)(\sin b + \cos b\,\delta b)\cos C.$$

Hence by subtraction, if we neglect the product $\delta a\, \delta b$,

$$0 = \delta a(\sin a \cos b - \cos a \sin b \cos \mathrm{C})$$
$$+ \delta b(\sin b \cos a - \cos b \sin a \cos \mathrm{C}) \; ; \qquad (1)$$

this gives the ratio of δa to δb in terms of a, b, C. We may express the ratio more simply in terms of A and B by use of formulae of the type of (19), Art. 52; for we thus get

$$\delta a \sin c \cos \mathrm{B} + \delta b \sin c \cos \mathrm{A} = 0,$$

or

$$\delta a \cos \mathrm{B} + \delta b \cos \mathrm{A} = 0. \qquad (2)$$

(ii) Required the connexion between the small variations of the other angles. In this case we may, by means of the polar triangle, deduce from the result just found that

$$\delta \mathrm{A} \cos b + \delta \mathrm{B} \cos a = 0 \; ; \qquad (3)$$

this may also be found independently as before.

(iii) Required the connexion between the small variations of a side and the opposite angle (A, a).

Here

$$\sin \mathrm{A} \sin c = \sin \mathrm{C} \sin a,$$

and

$$\sin(\mathrm{A} + \delta \mathrm{A}) \sin c = \sin \mathrm{C} \sin(a + \delta a) \; ;$$

hence by subtraction

$$\cos \mathrm{A} \sin c\, \delta \mathrm{A} = \sin \mathrm{C} \cos a\, \delta a,$$

and therefore

$$\delta \mathrm{A} \cot \mathrm{A} = \delta a \cot a. \qquad (4)$$

(iv) Required the connexion between the small variations of a side and the adjacent angle (a, B).

We have $\cot \mathrm{C} \sin \mathrm{B} = \cot c \sin a - \cos \mathrm{B} \cos a$;

proceeding as before, we obtain

$$\cot \mathrm{C} \cos \mathrm{B}\, \delta \mathrm{B} = \cot c \cos a\, \delta a + \cos \mathrm{B} \sin a\, \delta a + \cos a \sin \mathrm{B}\, \delta \mathrm{B} \; ;$$

therefore

$$(\cot \mathrm{C} \cos \mathrm{B} - \cos a \sin \mathrm{B})\, \delta \mathrm{B} = (\cot c \cos a + \cos \mathrm{B} \sin a)\, \delta a \; ;$$

therefore

$$-\frac{\cos \mathrm{A}}{\sin \mathrm{C}}\, \delta \mathrm{B} = \frac{\cos b}{\sin c}\, \delta a \; ;$$

therefore

$$\delta \mathrm{B} \cos \mathrm{A} = -\delta a \cot b \sin \mathrm{B}. \qquad (5)$$

This result may be obtained more briefly by eliminating $\delta \mathrm{A}$ from the relations (3) and (4).

244. When all the elements of a spherical triangle undergo slight changes, to find relations connecting these small variations.

This is the most general problem; we may derive from it results suitable to any particular problem by substituting zero for the variations of those elements which remain constant.

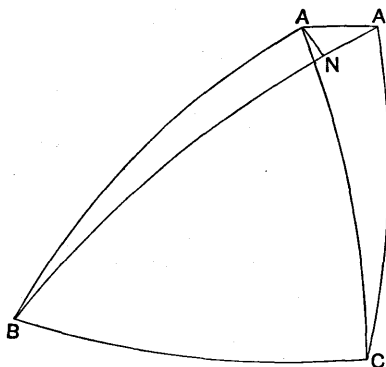
Let us take the three equalities of the type

$$\cos a = \cos b \cos c + \sin b \sin c \cos A, \qquad (6)$$

and treat them by the method exemplified in the previous article, allowing however for variation of *all* the elements. We thus get the following equations:

$$\left.\begin{array}{r} \delta a - \cos C\, \delta b - \cos B\, \delta c = \sin b \sin C\, \delta A, \\ -\cos C\, \delta a + \delta b - \cos A\, \delta c = \sin c \sin A\, \delta B, \\ -\cos B\, \delta a - \cos A\, \delta b + \delta c = \sin a \sin B\, \delta C. \end{array}\right\} \qquad (7)$$

These relations may be regarded as fundamental; from them it is easy to derive by elimination a relation between the variations of any four of the elements of the triangle.

245. Geometrical method. The geometrical method of evaluating small variations is useful (particularly in astronomical problems) and instructive; we shall illustrate it by an example.

The side c of a spherical triangle undergoes a small variation δc,

while the lengths of the other sides remain unaltered; it is required to find the changes produced in the angles A, B, *and* C.

Let ABC be the triangle having the sides a, b, c, and A′BC the triangle on the same base BC, having the sides a, b, $c+\delta c$. Then the angles of the former triangle being A, B, C, those of the latter will be $A+\delta A$, $B+\delta B$, $C+\delta C$, so that $A\hat{C}A' = \delta C$, and $A\hat{B}A' = -\delta B$.

Draw the arc AN perpendicular to BA′. In the right-angled triangle ANB $\quad \sin AN = \sin AB \sin ABN,$

and $\quad \tan BN = \tan AB \cos ABN;$

and consequently, when small quantities of the order of δB^2 are neglected, $\quad AN = -\delta B \sin c,$

$$BN = c,$$

and therefore also $\quad NA' = BA' - BN = \delta c.$

In a similar manner, since $CA = CA'$, it is readily seen, by drawing the arc through C perpendicular to and bisecting AA′, that $\quad AA' = \delta C \sin b.$

Thus the great-circle arc AA′ is very approximately coincident with the corresponding arc of the small circle having centre C and radius CA; and therefore the angles CAA′ and CA′A are approximately right angles. And so, to a first approximation,

$$N\hat{A}A' = A.$$

The triangle ANA′ has its sides so small that it may be regarded as a plane triangle, so that

$$AN = NA' \cot A, \quad AA' = NA' \operatorname{cosec} A.$$

Substituting in these results the values of AN, NA′, and AA′ obtained above, we get

$$\left.\begin{aligned} \delta B &= -\delta c \cot A \operatorname{cosec} c, \\ \text{and} \quad \delta C &= \delta c \operatorname{cosec} A \operatorname{cosec} b; \\ \text{while, from analogy with } \delta B, & \\ \delta A &= -\delta c \cot B \operatorname{cosec} c. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

These agree with the results found by putting $\delta a = 0$, $\delta b = 0$ in (7).

EXAMPLES XV.

1. In a spherical triangle, if C and c remain constant while a and b receive the small increments δa and δb respectively, shew that

$$\frac{\delta a}{\sqrt{(1-n^2\sin^2 a)}}+\frac{\delta b}{\sqrt{(1-n^2\sin^2 b)}}=0, \text{ where } n=\frac{\sin C}{\sin c}.$$

2. If C and c remain constant, and a small change be made in a, find the consequent changes in the other parts of the triangle. Find also the change in the area.

3. Supposing A and c to remain constant, prove the following equations connecting the small variations of pairs of the other elements:

$$\sin C\,\delta b=\sin a\,\delta B, \quad \delta b\sin C=-\delta C\tan a, \quad \delta a\tan C=\delta B\sin a,$$
$$\delta a\tan C=-\delta C\tan a, \quad \delta B\cos C=\delta a, \quad \delta B\cos a=-\delta C.$$

4. Supposing b and c to remain constant, prove the following equations connecting the small variations of pairs of the other elements:

$$\delta B\tan C=\delta C\tan B, \quad \delta a\cot C=-\delta B\sin a,$$
$$\delta a=\delta A\sin c\sin B, \quad \delta A\sin B\cos C=-\delta B\sin A.$$

5. Supposing B and C to remain constant, prove the following equations connecting the small variations of pairs of the other elements:

$$\delta b\tan c=\delta c\tan b, \quad \delta A\cot c=\delta b\sin A,$$
$$\delta A=\delta a\sin b\sin C, \quad \delta a\sin B\cos c=\delta b\sin A.$$

6. If A and C are constant, and b be increased by a small quantity, shew that a will be increased or diminished according as c is less or greater than a quadrant.

CHAPTER XV.

ON THE CONNEXION OF FORMULAE IN PLANE AND SPHERICAL TRIGONOMETRY.

246. The student must have perceived that many of the results obtained in *Spherical* Trigonometry resemble others with which he is familiar in *Plane* Trigonometry. This resemblance has been abundantly exemplified in the preceding pages, especially in Chapters VIII. and IX. We shall now shew how we may deduce formulae in Plane Trigonometry from formulae in Spherical Trigonometry; and shall investigate a few more theorems in Spherical Geometry, which are of interest principally on account of their connexion with known results in Plane Geometry.

247. From any formula in Spherical Trigonometry involving the elements of a triangle, one of them being a side, it is required to deduce the corresponding formula in Plane Trigonometry.*

Let α, β, γ be the lengths of the sides of the triangle, r the radius of the sphere, so that $\frac{\alpha}{r}, \frac{\beta}{r}, \frac{\gamma}{r}$ are the circular measures of the sides of the triangle; expand the functions of $\frac{\alpha}{r}, \frac{\beta}{r}, \frac{\gamma}{r}$ which occur in any proposed formula in powers of $\frac{\alpha}{r}, \frac{\beta}{r}, \frac{\gamma}{r}$ respectively; then, if we suppose r to become indefinitely great, the limiting form of the proposed formula will be a relation in Plane Trigonometry.

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* LAGRANGE.

For example, in Art. 214, from the formula

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

we deduce

$$\cos A = \frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma} + \frac{\alpha^4 + \beta^4 + \gamma^4 - 2\beta^2\gamma^2 - 2\gamma^2\alpha^2 - 2\alpha^2\beta^2}{24\beta\gamma r^2} + \ldots\ldots;$$

now suppose r to become infinite; then ultimately

$$\cos A = \frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma};$$

and this is the expression for the cosine of the angle of a plane triangle in terms of the sides.

Again, in Art. 220, from the formula

$$\frac{\sin A}{\sin B} = \frac{\sin a}{\sin b}$$

we deduce

$$\frac{\sin A}{\sin B} = \frac{\alpha}{\beta} + \frac{\alpha(\beta^2 - \alpha^2)}{6\beta r^2} + \ldots\ldots,$$

now suppose r to become infinite; then ultimately

$$\frac{\sin A}{\sin B} = \frac{\alpha}{\beta};$$

that is, in a plane triangle the sides are as the sines of the opposite angles.

248. To find the equation to a small circle of the sphere.

Let O be the pole of a small circle, S a fixed point on the

sphere, SX a fixed great circle of the sphere. Let $OS = \alpha$, $OSX = \beta$; then the position of O is determined by means of these angular coordinates α and β. Let P be any point on the circumference of the small circle, $PS = \theta$, $PSX = \phi$, so that θ and ϕ are the angular coordinates of P. Let $OP = r$. Then from the triangle OSP

$$\cos r = \cos \alpha \cos \theta + \sin \alpha \sin \theta \cos (\phi - \beta); \quad \text{.........(1)}$$

this gives a relation between the angular coordinates of any point on the circumference of the circle.

If the circle be a great circle then $r = \frac{1}{2}\pi$; thus the equation becomes

$$0 = \cos \alpha \cos \theta + \sin \alpha \sin \theta \cos (\phi - \beta). \quad \text{............(2)}$$

It will be observed that, in the particular case in which the sphere is the Earth, and S the north or south pole, the angular coordinates here used are readily expressible in terms of the *latitude* and *longitude* which serve to determine the positions of places on the Earth's surface; θ is the *complement of the latitude* and ϕ is the *longitude.*

249. Equation (1) of the preceding Article may be written thus:

$$\cos r (\cos^2 \tfrac{1}{2}\theta + \sin^2 \tfrac{1}{2}\theta)$$
$$= \cos \alpha (\cos^2 \tfrac{1}{2}\theta - \sin^2 \tfrac{1}{2}\theta) + 2 \sin \alpha \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta \cos (\phi - \beta).$$

Divide by $\cos^2 \frac{1}{2}\theta$ and rearrange; hence

$$\tan^2 \tfrac{1}{2}\theta (\cos r + \cos \alpha) - 2 \tan \tfrac{1}{2}\theta \sin \alpha \cos (\phi - \beta) + \cos r - \cos \alpha = 0.$$

Let $\tan \frac{1}{2}\theta_1$ and $\tan \frac{1}{2}\theta_2$ denote the values of $\tan \frac{1}{2}\theta$ found from this quadratic equation; then

$$\tan \frac{\theta_1}{2} \tan \frac{\theta_2}{2} = \frac{\cos r - \cos \alpha}{\cos r + \cos \alpha} = \tan \frac{\alpha + r}{2} \tan \frac{\alpha - r}{2}.$$

Thus the value of the product $\tan \frac{1}{2}\theta_1 \tan \frac{1}{2}\theta_2$ is *independent of* ϕ; this result corresponds to the well-known property of a circle in Plane Geometry which is demonstrated in Euclid III, 36. (Cf. Art. 170.)

250. To find the locus of the vertex of a spherical triangle of given base and area. (Cf. Art. 153.)

Let AB be the given base (of length c), $\mathrm{AC}=\theta$, $\mathrm{BAC}=\phi$. Since the area is given the spherical excess is known; denote it by E; then by Art. 132, (6) and (7),

$$\sin(\phi-\tfrac{1}{2}\mathrm{E})=\cot\tfrac{1}{2}\theta\cot\tfrac{1}{2}c\sin\tfrac{1}{2}\mathrm{E},$$

therefore $\quad 2\cot\tfrac{1}{2}c\sin\tfrac{1}{2}\mathrm{E}\cos^2\tfrac{1}{2}\theta=\sin\theta\sin(\phi-\tfrac{1}{2}\mathrm{E}),$

therefore

$$\cos\theta\cot\tfrac{1}{2}c\sin\tfrac{1}{2}\mathrm{E}+\sin\theta\cos(\phi-\tfrac{1}{2}\mathrm{E}+\tfrac{1}{2}\pi)=-\cot\tfrac{1}{2}c\sin\tfrac{1}{2}\mathrm{E}.$$

Comparing this with equation (1) of Art. 248, we see that the required locus is a circle. If we call a, β the angular coordinates of its pole, we have

$$\tan a=\frac{1}{\cot\tfrac{1}{2}c\sin\tfrac{1}{2}\mathrm{E}}=\frac{\tan\tfrac{1}{2}c}{\sin\tfrac{1}{2}\mathrm{E}},$$

$$\beta=\tfrac{1}{2}\mathrm{E}-\tfrac{1}{2}\pi.$$

It may be presumed from symmetry that the pole of this circle is in the great circle which bisects AB at right angles; and this presumption is easily verified. For the equation to that great circle is

$$0=\cos\theta\cos(\tfrac{1}{2}\pi-\tfrac{1}{2}c)+\sin\theta\sin(\tfrac{1}{2}\pi-\tfrac{1}{2}c)\cos(\phi-\pi),$$

and the values $\theta=a$, $\phi=\beta$ satisfy this equation.

251. To find the angular distance between the poles of the inscribed and circumscribed circles of a triangle.

Let P denote the pole of the inscribed circle, and Q the pole of the circumscribed circle of a triangle ABC; then $\mathrm{PAB}=\tfrac{1}{2}\mathrm{A}$, by Art. 119, and $\mathrm{QAB}=\mathrm{S}-\mathrm{C}$, by Art. 122; hence

$$\cos\mathrm{PAQ}=\cos\tfrac{1}{2}(\mathrm{B}-\mathrm{C});$$

and $\quad \cos\mathrm{PQ}=\cos\mathrm{PA}\cos\mathrm{QA}+\sin\mathrm{PA}\sin\mathrm{QA}\cos\tfrac{1}{2}(\mathrm{B}-\mathrm{C}).$

Now, by Art. 73 (see the figure of Art. 119),

$$\cos \mathrm{PA} = \cos \mathrm{PE} \cos \mathrm{AE} = \cos r \cos (s-a),$$

$$\sin \mathrm{PA} = \frac{\sin \mathrm{PE}}{\sin \mathrm{PAE}} = \frac{\sin r}{\sin \frac{1}{2}\mathrm{A}};$$

thus

$$\cos \mathrm{PQ} = \cos \mathrm{R} \cos r \cos (s-a) + \sin \mathrm{R} \sin r \cos \tfrac{1}{2}(\mathrm{B}-\mathrm{C}) \operatorname{cosec} \tfrac{1}{2}\mathrm{A}.$$

Therefore, by Art. 63,

$$\cos \mathrm{PQ} = \cos \mathrm{R} \cos r \cos (s-a) + \sin \mathrm{R} \sin r \sin \tfrac{1}{2}(b+c) \operatorname{cosec} \tfrac{1}{2}a,$$

therefore

$$\frac{\cos \mathrm{PQ}}{\cos \mathrm{R} \sin r} = \cot r \cos (s-a) + \tan R \sin \tfrac{1}{2}(b+c) \operatorname{cosec} \tfrac{1}{2}a.$$

Now $\qquad \cot r = \dfrac{\sin s}{n}, \quad \tan \mathrm{R} = \dfrac{2 \sin \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c}{n},$

therefore

$$\frac{\cos \mathrm{PQ}}{\cos \mathrm{R} \sin r} = \frac{1}{n}\{\sin s \cos (s-a) + 2 \sin \tfrac{1}{2}(b+c) \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c\}$$

$$= \frac{1}{2n}(\sin a + \sin b + \sin c).$$

Hence $\qquad \left(\dfrac{\cos \mathrm{PQ}}{\cos \mathrm{R} \sin r}\right)^2 - 1 = \dfrac{1}{4n^2}(\sin a + \sin b + \sin c)^2 - 1$

$$= (\cot r + \tan \mathrm{R})^2 \quad \text{(by Art. 124)};$$

therefore $\qquad \cos^2 \mathrm{PQ} = \cos^2 \mathrm{R} \sin^2 r + \cos^2 (\mathrm{R}-r),$

and $\qquad \sin^2 \mathrm{PQ} = \sin^2 (\mathrm{R}-r) - \cos^2 \mathrm{R} \sin^2 r.$

The limiting form of this equality, when the triangle is plane, is clearly

$$\mathrm{PQ}^2 = (\mathrm{R}-r)^2 - r^2,$$

$$= \mathrm{R}^2 - 2\mathrm{R}r,$$

a well known result.

252. To find the angular distance between the pole of the circumscribed circle and the pole of one of the escribed circles of a triangle.

Let Q denote the pole of the circumscribed circle, and Q_1

the pole of the escribed circle opposite to the angle A. Then it may be shewn that $QBQ_1 = \frac{1}{2}\pi + \frac{1}{2}(C - A)$, and

$$\cos QQ_1 = \cos R \cos r_1 \cos(s-c) - \sin R \sin r_1 \sin \tfrac{1}{2}(C-A) \sec \tfrac{1}{2}B$$
$$= \cos R \cos r_1 \cos(s-c) - \sin R \sin r_1 \sin \tfrac{1}{2}(c-a) \operatorname{cosec} \tfrac{1}{2}b.$$

Therefore

$$\frac{\cos QQ_1}{\sin r_1 \cos R} = \cot r_1 \cos(s-c) - \tan R \sin \tfrac{1}{2}(c-a) \operatorname{cosec} \tfrac{1}{2}b;$$

by reducing as in the preceding Article, the right-hand member of the last equation becomes

$$\frac{1}{2n}(\sin b + \sin c - \sin a);$$

hence $\left(\dfrac{\cos QQ_1}{\cos R \sin r_1}\right)^2 - 1 = (\tan R - \cot r_1)^2$, (Art. 124);

therefore $\cos^2 QQ_1 = \cos^2 R \sin^2 r_1 + \cos^2(R + r_1)$,

and $\sin^2 QQ_1 = \sin^2(R + r_1) - \cos^2 R \sin^2 r_1$.

For a plane triangle

$$QQ_1^2 = (R + r_1)^2 - r_1^2.$$

EXAMPLES XVI.

1. From the formula $\sin\dfrac{a}{2} = \sqrt{\left\{\dfrac{-\cos S \cos(S-A)}{\sin B \sin C}\right\}}$ deduce the expression for the area of a plane triangle, namely $\dfrac{a^2 \sin B \sin C}{2 \sin A}$, when the radius of the sphere is indefinitely increased.

2. Two triangles ABC, *abc*, spherical or plane, equal in all respects, differ slightly in position: shew that

$$\cos ABb \cos BCc \cos CAa + \cos ACc \cos CBb \cos BAa = 0.$$

3. Deduce formulae in Plane Trigonometry from NAPIER'S analogies.

4. Deduce formulae in Plane Trigonometry from DELAMBRE'S analogies.

5. From the formula $\cos\frac{1}{2}c \cos\frac{1}{2}(A+B) = \sin\frac{1}{2}C \cos\frac{1}{2}(a+b)$ deduce the area of a plane triangle in terms of the sides and one of the angles.

6. What result is obtained from Example VI, 7, by supposing the radius of the sphere infinite?

7. If one angle of a spherical triangle remains constant while the adjacent sides are increased, shew that the area and the sum of the angles are increased.

8. If the arcs bisecting two angles of a spherical triangle and terminated at the opposite sides are equal, the bisected angles will be equal provided their sum be less than 180°.

[Let BOD and COE denote these two arcs which are given equal. If the angles B and C are not equal suppose B the greater. Then CD is greater than BE by Art. 67. And as the angle OBC is greater than the angle OCB, therefore OC is greater than OB; therefore OD is greater than OE. Hence the angle ODC is greater than the angle OEB, by Example 7. Then construct a spherical triangle BCF on the other side of BC, equal to CBE. Since the angle ODC is greater than the angle OEB, the angle FDC is greater than the angle DFC; therefore CD is less than CF, so that CD is less than BE. See the corresponding problem in Plane Geometry in the *Appendix to Euclid*, page 317.]

CHAPTER XVI.

POLYHEDRONS.

253. A polyhedron is a solid bounded by any number of plane rectilineal figures which are called its faces. A polyhedron is said to be *regular* when its faces are similar and equal regular polygons, and its solid angles equal to one another.

254. *If* S *be the number of solid angles in any polyhedron,* F *the number of its faces,* E *the number of its edges, then* $S + F = E + 2$.

Take any point within the polyhedron as centre, and describe a sphere of radius r, and draw straight lines from the centre to each of the angular points of the polyhedron; let the points at which these straight lines meet the surface of the sphere be joined by arcs of great circles, so that the surface of the sphere is divided into as many polygons as the polyhedron has faces.

Let s denote the sum of the angles of any one of these polygons, m the number of its sides; then the area of the polygon is $r^2\{s - (m-2)\pi\}$ by Art. 129. The sum of the areas of all the polygons is the surface of the sphere, that is, $4\pi r^2$. Hence, since the number of the polygons is F, we obtain

$$4\pi = \Sigma s - \pi\Sigma m + 2F\pi.$$

Now Σs denotes the sum of all the angles of the polygons, and is therefore equal to $2\pi \times$ the number of solid angles, that

is, to $2\pi S$; and Σm is equal to the number of all the sides of all the polygons, that is, to $2E$, since every edge gives rise to an arc which is common to two polygons. Therefore

$$4\pi = 2\pi S - 2\pi E + 2F\pi;$$

therefore
$$S + F = E + 2.$$

255. *There can be only five regular polyhedrons.*

Let m be the number of sides in each face of a regular polyhedron, n the number of plane angles in each solid angle; then the entire number of plane angles is expressed by mF, or by nS, or by $2E$; thus

$$mF = nS = 2E, \text{ and } S + F = E + 2;$$

from these equations we obtain

$$S = \frac{4m}{2(m+n) - mn}, \quad E = \frac{2mn}{2(m+n) - mn}, \quad F = \frac{4n}{2(m+n) - mn}.$$

These expressions must be positive integers, we must therefore have $2(m+n)$ greater than mn; therefore

$$\frac{1}{m} + \frac{1}{n} \text{ must be greater than } \frac{1}{2};$$

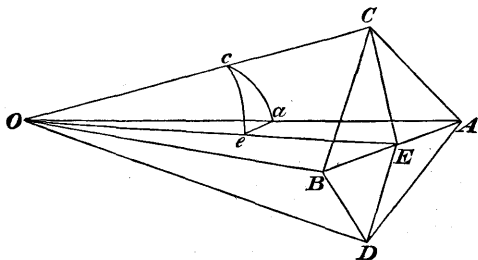
but n cannot be less than 3, so that $\frac{1}{n}$ cannot be greater than $\frac{1}{3}$, and therefore $\frac{1}{m}$ must be greater than $\frac{1}{6}$; and as m must be an integer and cannot be less than 3, the only admissible values of m are 3, 4, 5. It will be found on trial that the only values of m and n which satisfy all the necessary conditions are the following: each regular polyhedron derives its name from the number of its plane faces.

m	n	S	E	F	Name of Regular Polyhedron.
3	3	4	6	4	Tetrahedron or regular Pyramid.
4	3	8	12	6	Hexahedron or Cube.
3	4	6	12	8	Octahedron.
5	3	20	30	12	Dodecahedron.
3	5	12	30	20	Icosahedron.

It will be seen that the demonstration establishes something more than the enunciation states; for it is not assumed that the faces are equilateral and equiangular and all equal. It is in fact demonstrated that, *there cannot be more than five solids each of which has all its faces with the same number of sides, and all its solid angles formed with the same number of plane angles.*

256. *The sum of all the plane angles which form the solid angles of any polyhedron is* $2(S-2)\pi$.

For if m denote the number of sides in any face of the polyhedron, the sum of the interior angles of that face is $(m-2)\pi$ by Euclid I, 32, Cor. 1. Hence the sum of all the interior angles of all the faces is $\Sigma(m-2)\pi$, that is $\Sigma m\pi - 2F\pi$, that is $2(E-F)\pi$, that is $2(S-2)\pi$.

257. *To find the inclination of two adjacent faces of a regular polyhedron.*

Let AB be the edge common to the two adjacent faces, C and D the centres of the faces; bisect AB at E, and join CE and DE; CE and DE will be perpendicular to AB, and the angle CED is the angle of inclination of the two adjacent faces; we shall denote it by i. In the plane containing CE and DE draw CO and DO at right angles to CE and DE respectively, and meeting at O; about O as centre describe a sphere

meeting OA, OC, OE at a, c, e respectively, so that cae forms a spherical triangle. Since AB is perpendicular to CE and DE, it is perpendicular to the plane CED, therefore the plane AOB which contains AB is perpendicular to the plane CED; hence the angle cea of the spherical triangle is a right angle. Let m be the number of sides in each face of the polyhedron, n the number of the plane angles which form each solid angle. Then the angle $ace = \text{ACE} = \frac{2\pi}{2m} = \frac{\pi}{m}$; and the angle cae is half one of the n equal angles formed on the sphere round a, that is, $cae = \frac{2\pi}{2n} = \frac{\pi}{n}$. From the right-angled triangle cae

$$\cos cae = \cos ce \sin ace,$$

that is,

$$\cos\frac{\pi}{n} = \cos\left(\frac{\pi}{2} - \frac{i}{2}\right)\sin\frac{\pi}{m};$$

therefore

$$\sin\frac{i}{2} = \frac{\cos\frac{\pi}{n}}{\sin\frac{\pi}{m}}.$$

258. *To find the radii of the inscribed and circumscribed spheres of a regular polyhedron.*

Let the edge $\text{AB} = a$, let $\text{OC} = r$ and $\text{OA} = \text{R}$, so that r is the radius of the inscribed sphere, and R is the radius of the circumscribed sphere. Then

$$\text{CE} = \text{AE} \cot \text{ACE} = \frac{a}{2}\cot\frac{\pi}{m},$$

$$r = \text{CE} \tan \text{CEO} = \text{CE} \tan\frac{i}{2} = \frac{a}{2}\cot\frac{\pi}{m}\tan\frac{i}{2};$$

also

$$r = \text{R} \cos a\text{O}c = \text{R} \cot eca \cot eac = \text{R} \cot\frac{\pi}{m}\cot\frac{\pi}{n};$$

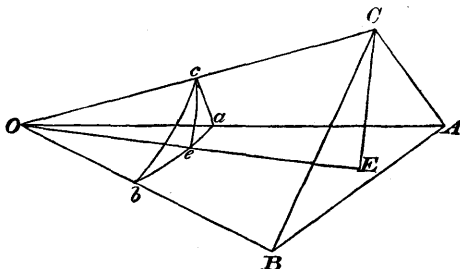
therefore

$$\text{R} = r\tan\frac{\pi}{m}\tan\frac{\pi}{n} = \frac{a}{2}\tan\frac{i}{2}\tan\frac{\pi}{n}.$$

259. *To find the surface and volume of a regular polyhedron.*

The area of one face of the polyhedron is $\frac{ma^2}{4}\cot\frac{\pi}{m}$, and therefore the surface of the polyhedron is $\frac{m\text{F}a^2}{4}\cot\frac{\pi}{m}$.

Also the volume of the pyramid which has one face of the polyhedron for base and O for vertex is $\frac{r}{3}\cdot\frac{ma^2}{4}\cot\frac{\pi}{m}$, and therefore the volume of the polyhedron is $\frac{m\text{F}ra^2}{12}\cot\frac{\pi}{m}$.

260. *To find the volume of a parallelepiped in terms of its edges and their inclinations to one another.*

Let the edges be OA $=a$, OB $=b$, OC $=c$; let the inclinations be BOC $=\alpha$, COA $=\beta$, AOB $=\gamma$. Draw CE perpendicular to the plane AOB meeting it at E. Describe a sphere with O as centre, meeting OA, OB, OC, OE at a, b, c, e respectively.

The volume of the parallelepiped is equal to the product of its base and altitude $= ab\sin\gamma\,.\,\text{CE} = abc\sin\gamma\sin cOe$. The spherical triangle cae is right-angled at e; thus

$$\sin ce = \sin ca \sin cae = \sin\beta \sin cab,$$

and from the spherical triangle cab (by Art. 45)

$$\sin cab = \frac{\sqrt{(1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma)}}{\sin\beta\sin\gamma};$$

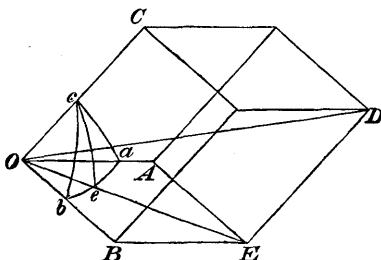
therefore the volume of the parallelepiped

$$= abc\sqrt{(1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma)}.$$

261. *To find the diagonal of a parallelepiped in terms of the three edges which it meets and their inclinations to one another.*

Let the edges be $OA = a$, $OB = b$, $OC = c$; let the inclinations be $BOC = \alpha$, $COA = \beta$, $AOB = \gamma$. Let OD be the diagonal required, and let OE be the diagonal of the face OAB. Then

$$OD^2 = OE^2 + ED^2 + 2OE.ED \cos COE$$
$$= a^2 + b^2 + 2ab \cos\gamma + c^2 + 2c.OE \cos COE.$$

Describe a sphere with O as centre meeting OA, OB, OC, OE at a, b, c, e respectively; then, by Art. 143, (11),

$$\cos COE = \cos ce = \frac{\cos cb \sin ae + \cos ca \sin be}{\sin ab}$$
$$= \frac{\cos\alpha \sin AOE + \cos\beta \sin BOE}{\sin\gamma};$$

therefore

$$OD^2 = a^2 + b^2 + c^2 + 2ab\cos\gamma + \frac{2c.OE}{\sin\gamma}(\cos\alpha \sin AOE + \cos\beta \sin BOE);$$

and $\quad OE \sin AOE = b \sin\gamma, \quad OE \sin BOE = a \sin\gamma,$

therefore $\quad OD^2 = a^2 + b^2 + c^2 + 2bc\cos\alpha + 2ca\cos\beta + 2ab\cos\gamma.$

262. *To find the volume of a tetrahedron.*

A tetrahedron is one-sixth of a parallelepiped which has the same altitude and its base double that of the tetrahedron; thus if the edges and their inclinations are given we can take one-sixth of the expression for the volume in Art. 260. The volume of a tetrahedron may also be expressed in terms of its

six edges; for in the figure of Art. 260 let $BC=a$, $CA=b$, $AB=c$, $OA=a'$, $OB=b'$, $OC=c'$; then

$$\cos\alpha=\frac{b'^2+c'^2-a^2}{2b'c'},\quad \cos\beta=\frac{c'^2+a'^2-b^2}{2c'a'},\quad \cos\gamma=\frac{a'^2+b'^2-c^2}{2a'b'};$$

if these values are substituted for $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ in the expression obtained in Art. 260, and the factor abc replaced by $a'b'c'$ in accordance with the altered notation, the volume of the tetrahedron will be expressed in terms of its six edges.

The following result will be obtained, in which V denotes the volume of the tetrahedron,

$$\begin{aligned}144\,V^2 = & -a^2b^2c^2\\ & +a^2a'^2(b^2+c^2-a^2)+b^2b'^2(c^2+a^2-b^2)+c^2c'^2(a^2+b^2-c^2)\\ & -a^2(a'^2-b'^2)(a'^2-c'^2)-b^2(b'^2-c'^2)(b'^2-a'^2)\\ & -c^2(c'^2-a'^2)(c'^2-b'^2).\end{aligned}$$

Thus for a *regular* tetrahedron we have $144\,V^2=2a^6$.

263. If the vertex of a tetrahedron be supposed to be situated at any point in the plane of its base, the volume vanishes; hence, if we equate to zero the expression on the right-hand side of the equation just given, we obtain a relation which must hold among the six straight lines which join four points taken arbitrarily in a plane.

Or we may adopt CARNOT'S method, in which this relation is established independently, and the expression for the volume of a tetrahedron is deduced from it; this we shall now shew, and we shall add some other investigations which are also given by CARNOT.

264. *To find the relation holding among the lengths of the six straight lines which join four points taken arbitrarily in a plane.*

Let A, B, C, D be the four points. Let $BC=a$, $CA=b$, $AB=c$; also let $DA=a'$, $DB=b'$, $DC=c'$.

If D falls *within* the triangle ABC, the sum of the angles BDC, CDA, ADB is equal to four right angles; these angles being denoted by θ, ϕ, ψ, and their sum by 2σ, it follows that $\sin\sigma=0$.

If D falls *without* the triangle ABC, one of the three angles at D is equal to the sum of the other two, that is to say, one of the quantities $\sigma - \theta$, $\sigma - \phi$, $\sigma - \psi$, vanishes.

Therefore, whether D be inside or outside,

$$0 = 4 \sin\sigma \sin(\sigma - \theta) \sin(\sigma - \phi) \sin(\sigma - \psi)$$
$$= 1 - \cos^2\theta - \cos^2\phi - \cos^2\psi + 2\cos\theta\cos\phi\cos\psi.$$

Now $\cos\theta = (b'^2 + c'^2 - a^2)/2b'c'$, and the other cosines may be expressed in a similar manner; substitute these values in the above result, and we obtain the required relation, which after reduction may be exhibited thus,

$$\begin{aligned} 0 = & -a^2b^2c^2 \\ & + a'^2a^2(b^2 + c^2 - a^2) + b'^2b^2(c^2 + a^2 - b^2) + c'^2c^2(a^2 + b^2 - c^2) \\ & - a^2(a'^2 - b'^2)(a'^2 - c'^2) - b^2(b'^2 - c'^2)(b'^2 - a'^2) \\ & - c^2(c'^2 - a'^2)(c'^2 - b'^2). \end{aligned}$$

265. *To express the volume of a tetrahedron in terms of the lengths of its six edges.*

Let a, b, c be the lengths of the sides of a triangle ABC forming one face of the tetrahedron, which we may call its base; let a', b', c' be the lengths of the straight lines which join A, B, C respectively to the vertex of the tetrahedron. Let p be the length of the perpendicular from the vertex on the base; then the lengths of the straight lines drawn from the foot of the perpendicular to A, B, C respectively are $\sqrt{(a'^2 - p^2)}$, $\sqrt{(b'^2 - p^2)}$, $\sqrt{(c'^2 - p^2)}$. Hence the relation given in Art. 264 will hold if we put $\sqrt{(a'^2 - p^2)}$ instead of a', $\sqrt{(b'^2 - p^2)}$ instead of b', and $\sqrt{(c'^2 - p^2)}$ instead of c'. We shall thus obtain

$$\begin{aligned} p^2(2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4) = & -a^2b^2c^2 \\ & + a'^2a^2(b^2 + c^2 - a^2) + b'^2b^2(c^2 + a^2 - b^2) + c'^2c^2(a^2 + b^2 - c^2) \\ & - a^2(a'^2 - b'^2)(a'^2 - c'^2) - b^2(b'^2 - c'^2)(b'^2 - a'^2) \\ & - c^2(c'^2 - a'^2)(c'^2 - b'^2). \end{aligned}$$

The coefficient of p^2 in this equation is sixteen times the square of the area of the triangle ABC; so that the left-hand

member is $144V^2$, where V denotes the volume of the tetrahedron. Hence the required expression is obtained.

266. *To find the relation holding among the six arcs of great circles which join four points taken arbitrarily on the surface of a sphere.* (Compare Art. 352, below.)

Let A, B, C, D be the four points. Let $BC = \alpha$, $CA = \beta$, $AB = \gamma$; let $DA = \alpha'$, $DB = \beta'$, $DC = \gamma'$.

As in Art. 264 we have

$$1 = \cos^2 BDC + \cos^2 CDA + \cos^2 ADB - 2 \cos BDC \cos CDA \cos ADB.$$

Now $\cos BDC = \dfrac{\cos\alpha - \cos\beta' \cos\gamma'}{\sin\beta' \sin\gamma'}$, and the other cosines may be expressed in a similar manner; substitute these values in the above result, and we obtain the required relation, which after reduction may be exhibited thus,

$$\begin{aligned}1 = {} & \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\alpha' + \cos^2\beta' + \cos^2\gamma' \\ & - \cos^2\alpha \cos^2\alpha' - \cos^2\beta \cos^2\beta' - \cos^2\gamma \cos^2\gamma' \\ & - 2 (\cos\alpha \cos\beta \cos\gamma + \cos\alpha \cos\beta' \cos\gamma' \\ & + \cos\beta \cos\gamma' \cos\alpha' + \cos\gamma \cos\alpha' \cos\beta') \\ & + 2 (\cos\beta \cos\gamma \cos\beta' \cos\gamma' + \cos\gamma \cos\alpha \cos\gamma' \cos\alpha' \\ & + \cos\alpha \cos\beta \cos\alpha' \cos\beta').\end{aligned}$$

267. *To find the radius of the sphere circumscribing a tetrahedron.*

Denote the edges of the tetrahedron as in Art. 265. Let the sphere be supposed to be circumscribed about the tetrahedron, and draw on the sphere the six arcs of great circles joining the angular points of the tetrahedron. Then the relation given in Art. 266 holds among the cosines of these six arcs.

Let r denote the radius of the sphere. Then

$$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2} = 1 - 2\left(\frac{a}{2r}\right)^2 = 1 - \frac{a^2}{2r^2};$$

and the other cosines may be expressed in a similar manner.

Substitute these values in the result of Art. 266, and we obtain, after reduction, with the aid of Art. 265,

$$4 \times 144\, V^2 r^2$$
$$= 2b^2c^2b'^2c'^2 + 2c^2a^2c'^2a'^2 + 2a^2b^2a'^2b'^2 - a^4a'^4 - b^4b'^4 - c^4c'^4.$$

The right-hand member may also be put into factors, as we see by recollecting the mode in which the expression for the area of a triangle is put into factors. Let $aa' + bb' + cc' = 2\sigma$; then

$$36\, V^2 r^2 = \sigma(\sigma - aa')(\sigma - bb')(\sigma - cc').$$

EXAMPLES XVII.

1. If i denote the inclination of two adjacent faces of a regular polyhedron, shew that $\cos i = \frac{1}{3}$ in the tetrahedron, $=0$ in the cube, $= -\frac{1}{3}$ in the octahedron, $= -\frac{1}{5}\sqrt{5}$ in the dodecahedron, and $= -\frac{1}{3}\sqrt{5}$ in the icosahedron.

2. With the notation of Art. 257, shew that the radius of the sphere which touches one face of a regular polyhedron and all the adjacent faces produced is $\frac{1}{2}a \cot \frac{\pi}{m} \cot \frac{1}{2}i$.

3. A sphere touches one face of a regular tetrahedron and the other three faces produced: find its radius.

4. If a and b are the radii of the spheres inscribed in and described about a regular tetrahedron, shew that $b = 3a$.

5. If a is the radius of a sphere inscribed in a regular tetrahedron, and R the radius of the sphere which touches the edges, shew that $R^2 = 3a^2$.

6. If a is the radius of a sphere inscribed in a regular tetrahedron, and R' the radius of the sphere which touches one face and the others produced, shew that $R' = 2a$.

7. If a cube and an octahedron be described about a given sphere, the sphere described about these polyhedrons will be the same; and conversely.

8. If a dodecahedron and an icosahedron be described about a given sphere, the sphere described about these polyhedrons will be the same; and conversely.

9. A regular tetrahedron and a regular octahedron are inscribed in the same sphere: compare the radii of the spheres which can be inscribed in the two solids.

10. The sum of the squares of the four diagonals of a parallelepiped is equal to four times the sum of the squares of the edges.

11. If with all the angular points of any parallelepiped as centres equal spheres be described, the sum of the intercepted portions of the parallelepiped will be equal in volume to one of the spheres.

12. A regular octahedron is inscribed in a cube so that the corners of the octahedron are at the centres of the faces of the cube : shew that the volume of the cube is six times that of the octahedron.

13. It is not possible to fill any given space with a number of regular polyhedrons of the same kind, except cubes; but this may be done by means of tetrahedrons and octahedrons which have equal faces, by using twice as many of the former as of the latter.

14. A spherical triangle is formed on the surface of a sphere of radius ρ; its angular points are joined, forming thus a pyramid with the straight lines joining them with the centre: shew that the volume of the pyramid is

$$\tfrac{1}{3}\rho^3\sqrt{(\tan r \tan r_1 \tan r_2 \tan r_3)},$$

where r, r_1, r_2, r_3 are the radii of the inscribed and escribed circles of the triangle.

15. The angular points of a regular tetrahedron inscribed in a sphere of radius r being taken as poles, four equal small circles of the sphere are described, so that each circle touches the other three. Shew that the area of the surface bounded by each circle is $2\pi r^2\left(1 - \frac{1}{\sqrt{3}}\right)$.

16. If O be any point within a spherical triangle ABC, the product of the sines of any two sides and the sine of the included angle

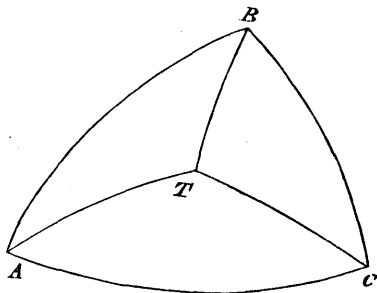
$$= \sin AO \sin BO \sin CO \{\cot AO \sin BOC + \cot BO \sin COA + \cot CO \sin AOB\}.$$

CHAPTER XVII.

ARCS DRAWN TO FIXED POINTS ON THE SURFACE OF A SPHERE.

268. In the present Chapter we shall demonstrate various propositions relating to the arcs drawn from any point on the surface of a sphere to certain fixed points on the surface.

269. ABC is a spherical triangle having all its sides quadrants, and therefore all its angles right angles; T is any point on the surface of the sphere: to shew that

$$\cos^2 TA + \cos^2 TB + \cos^2 TC = 1.$$

By Art. 42 we have

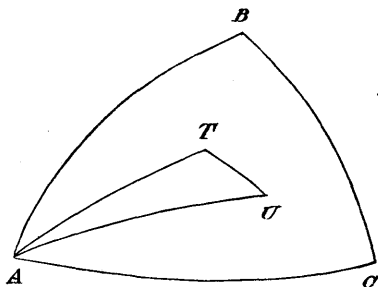
$$\cos TA = \cos AB \cos TB + \sin AB \sin TB \cos TBA$$
$$= \sin TB \cos TBA.$$

Similarly $\cos TC = \sin TB \cos TBC = \sin TB \sin TBA$.
Square and add; thus

$$\cos^2 TA + \cos^2 TC = \sin^2 TB = 1 - \cos^2 TB;$$

therefore $$\cos^2 TA + \cos^2 TB + \cos^2 TC = 1.$$

270. ABC is a spherical triangle having all its sides quadrants, and therefore all its angles right angles; T and U are any points on the surface of the sphere: to shew that

$$\cos TU = \cos TA \cos UA + \cos TB \cos UB + \cos TC \cos UC.$$

By Art. 42 we have

$$\cos TU = \cos TA \cos UA + \sin TA \sin UA \cos TAU;$$

now
$$\begin{aligned}\cos TAU &= \cos (BAU - BAT)\\ &= \cos BAU \cos BAT + \sin BAU \sin BAT\\ &= \cos BAU \cos BAT + \cos CAU \cos CAT;\end{aligned}$$

therefore
$$\cos TU = \cos TA \cos UA$$
$$+ \sin TA \sin UA (\cos BAU \cos BAT + \cos CAU \cos CAT);$$

and
$$\begin{aligned}\cos TB &= \sin TA \cos BAT,\\ \cos UB &= \sin UA \cos BAU,\\ \cos TC &= \sin TA \cos CAT,\\ \cos UC &= \sin UA \cos CAU;\end{aligned}$$

therefore

$$\cos TU = \cos TA \cos UA + \cos TB \cos UB + \cos TC \cos UC.$$

271. We leave to the student the exercise of shewing that the formulae of the two preceding Articles are perfectly general for all positions of T and U, outside or inside the triangle ABC: the demonstrations will remain essentially the same for all modifications of the diagrams. The formulae are of constant application in Analytical Geometry of three dimensions, and are demonstrated in works on that subject; we have given them here as they may be of service in Spherical Trigonometry, and will in fact now be used in obtaining some important results.

272. Let there be any number of fixed points on the surface of a sphere; denote them by H_1, H_2, H_3,.... Let T be any point on the surface of the sphere. We shall now investigate an expression for the sum of the cosines of the arcs which join T with the fixed points.

Denote the sum by Σ; so that

$$\Sigma = \cos TH_1 + \cos TH_2 + \cos TH_3 + \ldots$$

Take on the surface of the sphere a fixed spherical triangle ABC, having all its sides quadrants, and therefore all its angles right angles.

Let λ, μ, ν be the cosines of the arcs which join T with A, B, C respectively; let l_1, m_1, n_1 be the cosines of the arcs which join H_1 with A, B, C respectively; and let a similar notation be used with respect to H_2, H_3,...

Then, by Art. 270,

$$\Sigma = l_1\lambda + m_1\mu + n_1\nu + l_2\lambda + m_2\mu + n_2\nu + \ldots$$
$$= P\lambda + Q\mu + R\nu;$$

where P stands for $l_1 + l_2 + l_3 + \ldots$, with corresponding meanings for Q and R.

273. It will be seen that P is the value which Σ takes when T coincides with A, that Q is the value which Σ takes when T coincides with B, and that R is the value which Σ takes when

T coincides with C. Hence the result expresses the general value of Σ in terms of the cosines of the arcs which join T to the fixed points A, B, C, and the particular values of Σ which correspond to these three points.

274. We shall now transform the result of Art. 272.

Let
$$G = \sqrt{(P^2 + Q^2 + R^2)};$$
and let α, β, γ be three arcs determined by the equations
$$\cos\alpha = \frac{P}{G}, \quad \cos\beta = \frac{Q}{G}, \quad \cos\gamma = \frac{R}{G};$$
then
$$\Sigma = G(\lambda\cos\alpha + \mu\cos\beta + \nu\cos\gamma).$$

Since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, it is obvious that there will be some point on the surface of the sphere, such that α, β, γ are the arcs which join it to A, B, C respectively; denote this point by U; then by Art. 270,
$$\cos TU = \lambda\cos\alpha + \mu\cos\beta + \nu\cos\gamma;$$
and finally
$$\Sigma = G\cos TU.$$

Thus, whatever may be the position of T, the sum of the cosines of the arcs which join T to the fixed points varies as the cosine of the single arc which joins T to a certain fixed point U.

We might take G either positive or negative; it will be convenient to suppose it positive.

275. A sphere is described about a regular polyhedron; from any point on the surface of the sphere arcs are drawn to the solid angles of the polyhedron: to shew that the sum of the cosines of these arcs is zero.

From the preceding Article we see that if G is not zero there is *one* position of T which gives to Σ its greatest positive value, namely, when T coincides with U. But by the symmetry of a regular polyhedron there must always be *more than one*

positions of T which give the same value to Σ. For instance, if we take a regular tetrahedron, as there are *four* faces there will at least be *three other* positions of T symmetrical with any assigned position.

Hence G must be zero; and thus the *sum of the cosines of the arcs which join* T *to the solid angles of the regular polyhedron is zero for all positions of* T.

276. Since $G=0$, it follows that P, Q, R must each be zero; these indeed are particular cases of the general result of Art. 275. See Art. 273.

277. The result obtained in Art. 275 may be shewn to hold also in some other cases. Suppose, for instance, that a rectangular parallelepiped is inscribed in a sphere; then the sum of the cosines of the arcs drawn from any point on the surface of the sphere to the solid angles of the parallelepiped is zero. For here it is obvious that there must always be at least *one other* position of T symmetrical with any assigned position. Hence, by the argument of Art. 275, we must have $G=0$.

278. Let there be any number of fixed points on the surface of a sphere; denote them by H_1, H_2, H_3, ... Let T be any point on the surface of the sphere. We shall now investigate a remarkable expression for the sum of the squares of the cosines of the arcs which join T with the fixed points.

Denote the sum by Σ; so that

$$\Sigma = \cos^2 TH_1 + \cos^2 TH_2 + \cos^2 TH_3 + \dots.$$

Take on the surface of the sphere a fixed spherical triangle ABC, having all its sides quadrants, and therefore all its angles right angles.

Let λ, μ, ν be the cosines of the arcs which join T with A, B, C respectively; let l_1, m_1, n_1 be the cosines of the arcs which join H_1 with A, B, C respectively; and let a similar notation be used with respect to H_2, H_3, ...

Then, by Art. 270,

$$\Sigma = (l_1\lambda + m_1\mu + n_1\nu)^2 + (l_2\lambda + m_2\mu + n_2\nu)^2 + \dots$$

Expand each square, and rearrange the terms; thus

$$\Sigma = \mathsf{P}\lambda^2 + \mathsf{Q}\mu^2 + \mathsf{R}\nu^2 + 2p\mu\nu + 2q\nu\lambda + 2r\lambda\mu,$$

where P stands for $l_1{}^2 + l_2{}^2 + l_3{}^2 + \dots$,

and p stands for $m_1n_1 + m_2n_2 + m_3n_3 + \dots$,

with corresponding meanings for Q and q, and for R and r.

We shall shew that there is some position of the triangle ABC for which p, q, and r will vanish.

For let the triangle ABC be shifted to a new position A′B′C′, and let λ', μ', ν' be the cosines of the arcs TA′, TB′, TC′; then we have $\lambda' = \lambda \cos \mathrm{AA'} + \mu \cos \mathrm{BA'} + \nu \cos \mathrm{CA'}$,

with similar expressions for μ' and ν'; thus λ', μ', ν' are *linear* functions of λ, μ, ν, and are moreover such that

$$\lambda'^2 + \mu'^2 + \nu'^2 = 1 = \lambda^2 + \mu^2 + \nu^2.$$

Now, by a well known theorem of Algebra,* it is always possible to find a linear substitution by which the two quadratic forms Σ and $\lambda^2 + \mu^2 + \nu^2$ shall be reduced simultaneously to the forms

$$\Sigma = \mathfrak{P}\lambda'^2 + \mathfrak{Q}\mu'^2 + \mathfrak{R}\nu'^2,$$
$$\lambda^2 + \mu^2 + \nu^2 = \lambda'^2 + \mu'^2 + \nu'^2.$$

The constants $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$ are the roots of the cubic equation

$$\begin{vmatrix} \mathsf{P} - x, & r, & q \\ r, & \mathsf{Q} - x, & p \\ q, & p, & \mathsf{R} - x \end{vmatrix} = 0,$$

and an equality among the roots $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$ does not affect the result.

Therefore we see that there must be some position of the triangle ABC, such that for every position of T

$$\Sigma = \mathfrak{P}\lambda^2 + \mathfrak{Q}\mu^2 + \mathfrak{R}\nu^2.$$

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* CAUCHY (1829). The case when the cubic has equal roots is dealt with by WEIERSTRASS, *Berliner Monatsberichte*, 1858. A simple discussion of the theorem is given by Mr. T. J. I'A. BROMWICH in the *Quarterly Journal of Mathematics*, 1901.

279. The remarks of Art. 273 are applicable to the result just obtained.

280. Of the three constants $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$ determined in Art. 278, let $\mathfrak{P}$ be not less than the other two, and let $\mathfrak{R}$ be not greater than the other two. Then $\mathfrak{P}$ is readily seen to be the greatest value which Σ can receive, and $\mathfrak{R}$ the least value.

For, since by Art. 269,

$$\lambda^2 + \mu^2 + \nu^2 = 1,$$

$$\Sigma = \mathfrak{P}(1 - \mu^2 - \nu^2) + \mathfrak{Q}\mu^2 + \mathfrak{R}\nu^2$$
$$= \mathfrak{P} - (\mathfrak{P} - \mathfrak{Q})\mu^2 - (\mathfrak{P} - \mathfrak{R})\nu^2;$$

and

$$\Sigma = \mathfrak{P}\lambda^2 + \mathfrak{Q}\mu^2 + \mathfrak{R}(1 - \lambda^2 - \mu^2)$$
$$= \mathfrak{R} + (\mathfrak{P} - \mathfrak{R})\lambda^2 + (\mathfrak{Q} - \mathfrak{R})\mu^2.$$

Now, by supposition, none of the quantities $\mathfrak{P} - \mathfrak{Q}$, $\mathfrak{P} - \mathfrak{R}$, $\mathfrak{Q} - \mathfrak{R}$, can be negative; hence Σ cannot be greater than $\mathfrak{P}$ or less than $\mathfrak{R}$.

281. A sphere is described about a regular polyhedron; from any point on the surface of the sphere arcs are drawn to the solid angles of the polyhedron; it is required to find the sum of the squares of the cosines of these arcs.

With the notation of Art. 278 we have

$$\Sigma = \mathfrak{P}\lambda^2 + \mathfrak{Q}\mu^2 + \mathfrak{R}\nu^2.$$

We shall shew that in the present case $\mathfrak{P}$, $\mathfrak{Q}$, and $\mathfrak{R}$ must *all be equal.* For, if they are not, one of them must be greater than each of the others, or one of them must be less than each of the others.

If possible let the former be the case; suppose that $\mathfrak{P}$ is greater than $\mathfrak{Q}$, and greater than $\mathfrak{R}$. Then the equality

$$\Sigma = \mathfrak{P} - (\mathfrak{P} - \mathfrak{Q})\mu^2 - (\mathfrak{P} - \mathfrak{R})\nu^2,$$

shews that Σ is *always less than* $\mathfrak{P}$ except when $\mu = 0$ and $\nu = 0$: that is Σ is *always less than* $\mathfrak{P}$ except when T is at A, or at the point of the surface which is diametrically opposite to A.

But by the symmetry of a regular polyhedron there must always be *more than two* positions of T which give the same value to Σ. For instance if we take a regular tetrahedron, as there are *four* faces there will be at least *three other* positions of T symmetrical with any assigned position. Hence $\mathfrak{P}$ cannot be greater than $\mathfrak{Q}$ and greater than $\mathfrak{R}$.

In the same way we can shew that one of the three $\mathfrak{P}$, $\mathfrak{Q}$, and $\mathfrak{R}$, cannot be less than each of the others.

Therefore $\mathfrak{P} = \mathfrak{Q} = \mathfrak{R}$; and therefore by Art. 269 for *every position* of T we have $\Sigma = \mathfrak{P}$.

Since $\mathfrak{P} = \mathfrak{Q} = \mathfrak{R}$, each of them $= \frac{1}{3}(\mathfrak{P} + \mathfrak{Q} + \mathfrak{R})$

$$= \frac{1}{3}\{l_1^2 + m_1^2 + n_1^2 + l_2^2 + m_2^2 + n_2^2 + \dots\}$$

$$= \frac{S}{3}, \text{ by Art. 269,}$$

where S is the number of the solid angles of the regular polyhedron.

Thus the sum of the squares of the cosines of the arcs which join any point on the surface of the sphere to the solid angles of the regular polyhedron is one third of the number of the solid angles.

282. Since $\mathfrak{P} = \mathfrak{Q} = \mathfrak{R}$ in the preceding Article, it will follow that, when the fixed points of Art. 278 are the solid angles of a regular polyhedron, then for *any* position of the spherical triangle ABC we shall have $p = 0$, $q = 0$, $r = 0$.

For, taking any position for the spherical triangle ABC, we have $$\Sigma = P\lambda^2 + Q\mu^2 + R\nu^2 + 2p\mu\nu + 2q\nu\lambda + 2r\lambda\mu;$$
then at A we have $\mu = 0$ and $\nu = 0$, so that P is then the value of Σ; similarly Q and R are the values of Σ at B and C respectively. But by Art. 281 we have the same value for Σ whatever be the position of T; thus $\Sigma = P = Q = R$, and so

$$P = P(\lambda^2 + \mu^2 + \nu^2) + 2p\mu\nu + 2q\nu\lambda + 2r\lambda\mu;$$

therefore $$0 = 2p\mu\nu + 2q\nu\lambda + 2r\lambda\mu.$$

This holds then for every position of T. Suppose T is at *any point* of the great circle of which A is the pole; then $\lambda=0$: thus we get $p\mu\nu=0$ for all values of μ and ν, and therefore $p=0$. Similarly $q=0$, and $r=0$.

Expressed algebraically, the result here obtained is equivalent to the theorem* that, if the equation

$$\begin{vmatrix} \mathsf{P}-x, & r, & q \\ r, & \mathsf{Q}-x, & p \\ q, & p, & \mathsf{R}-x \end{vmatrix}=0$$

have its three roots equal, then must $\mathsf{P}=\mathsf{Q}=\mathsf{R}$, and $p=0$, $q=0$, $r=0$.

283. Let there be any number of fixed points on the surface of a sphere; denote them by H_1, H_2, H_3, ...; from any two points T and U on the surface of the sphere arcs are drawn to the fixed points: it is required to find the sum of the products of the corresponding cosines, that is

$$\cos \mathsf{TH}_1 \cos \mathsf{UH}_1 + \cos \mathsf{TH}_2 \cos \mathsf{UH}_2 + \cos \mathsf{TH}_3 \cos \mathsf{UH}_3 + \dots.$$

Let the notation be the same as in the beginning of Art. 278; and let λ', μ', ν be the cosines of the arcs which join U with A, B, C respectively. Then by Art. 270,

$$\begin{aligned}\cos \mathsf{TH}_1 \cos \mathsf{UH}_1 &= (\lambda l_1+\mu m_1+\nu n_1)(\lambda' l_1+\mu' m_1+\nu' n_1) \\ &= \lambda\lambda' l_1{}^2+\mu\mu' m_1{}^2+\nu\nu' n_1{}^2+(\mu\nu'+\nu\mu')m_1 n_1 \\ &\quad +(\nu\lambda'+\lambda\nu')n_1 l_1+(\lambda\mu'+\mu\lambda')l_1 m_1.\end{aligned}$$

Similar results hold for $\cos \mathsf{TH}_2 \cos \mathsf{UH}_2$, $\cos \mathsf{TH}_3 \cos \mathsf{UH}_3$, ... Hence, with the notation of Art. 278, the required sum is

$$\lambda\lambda'\mathsf{P}+\mu\mu'\mathsf{Q}+\nu\nu'\mathsf{R}+(\mu\nu'+\nu\mu')p+(\nu\lambda'+\lambda\nu')q+(\lambda\mu'+\mu\lambda')r.$$

Now by properly choosing the position of the triangle ABC we have p, q, and r each zero as in Art. 278; and thus the required sum becomes

$$\lambda\lambda'\mathfrak{P}+\mu\mu'\mathfrak{Q}+\nu\nu'\mathfrak{R}.$$

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* WEIERSTRASS, *loc. cit.*; BURNSIDE and PANTON, *Theory of Equations*, Chapter XIII, Examples 39 and 40.

284. The result obtained in Art. 278 may be considered as a particular case of that just given; namely the case in which the points T and U coincide.

285. A sphere is described about a regular polyhedron; from any two points on the surface of the sphere arcs are drawn to the solid angles of the polyhedron; it is required to find the sum of the products of the corresponding cosines.

With the notation of Art. 283 we see that the sum is

$$\lambda\lambda'\mathfrak{P} + \mu\mu'\mathfrak{Q} + \nu\nu'\mathfrak{R}.$$

And here $\mathfrak{P} = \mathfrak{Q} = \mathfrak{R} = \frac{S}{3}$, by Art. 281.

Thus the sum $= \frac{S}{3}(\lambda\lambda' + \mu\mu' + \nu\nu') = \frac{S}{3}\cos \mathrm{TU}$.

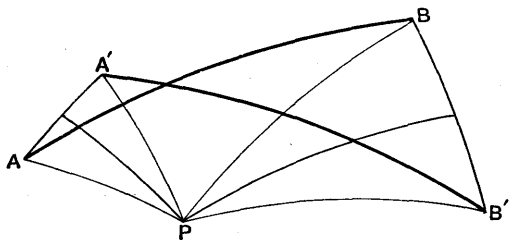
Thus the sum of the products of the cosines is equal to the product of the cosine of TU *into a third of the number of the solid angles of the regular polyhedron.*

286. The result obtained in Art. 281 may be considered as a particular case of that just given; namely, the case in which the points T and U coincide.

287. If TU is a quadrant then cos TU is zero, and the sum of the products of the cosines in Art. 285 is zero. The results $p = 0$, $q = 0$, $r = 0$, are easily seen to be all special examples of this particular case.

CHAPTER XVIII.

MISCELLANEOUS PROPOSITIONS.

288. *If* AB *and* A′B′ *be any two equal arcs, and the arcs* AA′ *and* BB′ *be bisected at right angles by arcs meeting at* P, *then the triangles* APB, A′PB′ *are identically equal to one another.**

For PA = PA′ and PB = PB′; hence the sides of the triangle PAB are respectively equal to those of PA′B′; therefore the triangles are identically equal; in particular the angle APB = the angle A′PB′, and therefore also APA′ = BPB′.

This simple proposition has an important application to the motion of a rigid body of which one point is fixed. For conceive a sphere capable of motion round its centre which is fixed; then it appears from this proposition that any two selected points on the sphere, as A and B, can be brought simultaneously into any other positions, as A′ and B′, by a rotation of the sphere round an axis passing through its centre

* Euler, *Theoria motus corporum solidorum*, 978.

and a certain point P. Hence it may be inferred that any change of position in a rigid body, of which one point is fixed, may be effected by a rotation round some axis through the fixed point.

289. Let P denote any point within any *plane* angle AOB, and from P draw perpendiculars on the straight lines OA and OB; then it is evident that these perpendiculars include an angle which is the *supplement* of the angle AOB. The corresponding fact with respect to a *solid* angle is worthy of notice. Let there be a solid angle formed by three plane angles, meeting at a point O. From any point P within the solid angle, draw perpendiculars PL, PM, PN on the three planes which form the solid angle; then the spherical triangle which corresponds to the three planes MPN, NPL, LPM is the *polar triangle* of the spherical triangle which corresponds to the solid angle at O. This remark is due to Professor DE MORGAN.

290. Suppose three straight lines to meet at a point and form a solid angle; let α, β, and γ denote the angles contained by these three straight lines taken in pairs: then it has been proposed to call the expression

$$\sqrt{(1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma)},$$

the *sine of the solid angle*. See BALTZER'S *Theorie ... der Determinanten*, 2nd edition, page 177. Adopting this definition it it is easy to shew that the sine of the solid angle lies between zero and unity. (Cp. Art. 51.)

We know that the area of a plane triangle is half the product of two sides into the sine of the included angle: by Art. 260 we have the following analogous proposition; the volume of a tetrahedron is one sixth of the product of three edges into the sine of the solid angle which they form.

Again, we know in mechanics that if three forces acting at a point are in equilibrium, each force is as the sine of the

angle between the directions of the other two: the following proposition is analogous; if four forces acting at a point are in equilibrium each force is as the sine of the solid angle formed by the directions of the other three. See *Statics*, Chapter II.

291. Let a sphere be described about a regular polyhedron; let perpendiculars be drawn from the centre of the sphere on the faces of the polyhedron, and produced to meet the surface of the sphere: then it is obvious from symmetry that the points of intersection must be the angular points of another regular polyhedron.

This may be verified. It will be found on examination that if S be the number of solid angles, and F the number of faces of one regular polyhedron, then another regular polyhedron exists which has S faces and F solid angles. See Art. 255.

292. Polyhedrons. The result in Art. 254 was first obtained by EULER; the demonstration which is there given is due to LEGENDRE. The demonstration shews that the result is true in many cases in which the polyhedron has *re-entrant* solid angles; for all that is necessary for the demonstration is that it shall be possible to take a point within the polyhedron as the centre of a sphere, so that the polygons, formed as in Art. 254, shall not have any coincident portions. The result, however, is generally true, even in cases in which the condition required by the demonstration of Art. 254 is not satisfied. We shall accordingly give another demonstration, and shall then deduce some important consequences from the result. We begin with a theorem which is due to CAUCHY.

293. *Let there be any network of rectilineal figures, not necessarily in one plane, but not forming a closed surface; let* E *be the number of edges,* F *the number of figures, and* S *the number of corner points: then* $F + S = E + 1$.

This theorem is obviously true in the case of a single plane figure; for then $\mathsf{F}=1$, and $\mathsf{S}=\mathsf{E}$. It can be shewn to be generally true by induction. For assume the theorem to be true for a network of F figures; and suppose that a rectilineal figure of n sides is added to this network, so that the network and the additional figure have m sides coincident, and therefore $m+1$ corner points coincident. And with respect to the new network which is thus formed, let E', F', S' denote the same things as E, F, S with respect to the old network. Then

$$\mathsf{E}'=\mathsf{E}+n-m, \quad \mathsf{F}'=\mathsf{F}+1, \quad \mathsf{S}'=\mathsf{S}+n-(m+1);$$

therefore $$\mathsf{F}'+\mathsf{S}'-\mathsf{E}'=\mathsf{F}+\mathsf{S}-\mathsf{E}.$$

But $\mathsf{F}+\mathsf{S}=\mathsf{E}+1$, by hypothesis; therefore $\mathsf{F}'+\mathsf{S}'=\mathsf{E}'+1$.

294. To demonstrate EULER'S theorem we suppose one face of a polyhedron removed, and we thus obtain a network of rectilineal figures to which CAUCHY'S theorem is applicable.

Thus $$\mathsf{F}-1+\mathsf{S}=\mathsf{E}+1;$$

therefore $$\mathsf{F}+\mathsf{S}=\mathsf{E}+2.$$

295. *In any polyhedron the number of faces with an odd number of sides is even, and the number of solid angles formed with an odd number of plane angles is even.*

Let a, b, c, d,...... denote respectively the numbers of faces which are triangles, quadrilaterals, pentagons, hexagons,...... Let α, β, γ, δ,...... denote respectively the numbers of the solid angles which are formed with three, four, five, six,...... plane angles.

Then, each edge belongs to *two* faces, and terminates at *two* solid angles; therefore

$$2\mathsf{E}=3a+4b+5c+6d+\ldots\ldots,$$

$$2\mathsf{E}=3\alpha+4\beta+5\gamma+6\delta+\ldots\ldots.$$

From these relations it follows that $a+c+e+\ldots\ldots$, and $\alpha+\gamma+\epsilon+\ldots\ldots$ are *even* numbers.

296. With the notation of the preceding Article we have

$$\text{F} = a + b + c + d + \ldots\ldots,$$
$$\text{S} = \alpha + \beta + \gamma + \delta + \ldots\ldots.$$

From these combined with the former relations we obtain

$$2\text{E} - 3\text{F} = b + 2c + 3d + \ldots\ldots,$$
$$2\text{E} - 3\text{S} = \beta + 2\gamma + 3\delta + \ldots\ldots.$$

Thus 2E cannot be less than 3F, or less than 3S.

297. From the expressions for E, F, and S, given in the two preceding Articles, combined with the result $2\text{F} + 2\text{S} = 4 + 2\text{E}$, we obtain

$$2(a+b+c+d+\ldots) + 2(\alpha+\beta+\gamma+\delta+\ldots)$$
$$= 4 + 3a + 4b + 5c + 6d + \ldots,$$

$$2(a+b+c+d+\ldots) + 2(\alpha+\beta+\gamma+\delta+\ldots)$$
$$= 4 + 3\alpha + 4\beta + 5\gamma + 6\delta + \ldots,$$

therefore

$$2(\alpha+\beta+\gamma+\delta+\ldots) - (a + 2b + 3c + 4d + \ldots) = 4 \ldots\ldots(1)$$
$$2(a+b+c+d+\ldots) - (\alpha + 2\beta + 3\gamma + 4\delta + \ldots) = 4 \ldots\ldots(2)$$

Therefore, by addition

$$a + \alpha - (c+\gamma) - 2(d+\delta) - 3(e+\epsilon) - \ldots\ldots = 8.$$

Thus the number of triangular faces together with the number of solid angles formed with three plane angles, cannot be less than eight.

Again, from (1) and (2), by eliminating α, we obtain

$$3a + 2b + c - e - 2f - \ldots\ldots - 2\beta - 4\gamma - \ldots\ldots = 12,$$

so that $3a + 2b + c$ cannot be less than 12. From this result various inferences can be drawn; thus, for example, *a solid cannot be formed which shall have no triangular, quadrilateral, or pentagonal faces.*

In like manner we can shew that $3\alpha + 2\beta + \gamma$ cannot be less than 12.

298. Poinsot has shewn that, in addition to the five well-known *regular polyhedrons*, four other solids exist which are perfectly symmetrical in shape, and which might therefore also be called *regular*. We may give an idea of the nature of Poinsot's results by referring to the case of a polygon. Suppose five points A, B, C, D, E, placed in succession at equal distances round the circumference of a circle. If we draw a straight line from each point to the next point, we form an ordinary regular pentagon. Suppose however we join the points by straight lines in the following order, A to C, C to E, E to B, B to D, D to A; we thus form a star-shaped symmetrical figure, which might be considered a regular pentagon.

It appears that, in a like manner, four, and only four, new regular solids can be formed. To such solids, the faces of which intersect and cross, Euler's theorem does not apply.

299. Let us return to Art. 293, and suppose e the number of edges *in* the bounding contour, and e' the number of edges *within* it; also suppose s the number of corners *in* the bounding contour, and s' the number *within* it. Then

$$E = e + e'; \quad S = s + s';$$

therefore $$1 + e + e' = s + s' + F.$$

But $$e = s;$$

therefore $$1 + e' = s' + F.$$

We can now demonstrate an extension of Euler's theorem, which has been given by Cauchy.

300. *Let a polyhedron be decomposed into any number of polyhedrons at pleasure; let* P *be the number thus formed,* S *the number of solid angles,* F *the number of faces,* E *the number of edges: then*

$$S + F = E + P + 1.$$

For suppose all the polyhedrons united, by starting with one and adding one at a time. Let e, f, s be respectively the

numbers of edges, faces, and solid angles in the first; let e', f', s' be respectively the numbers of edges, faces, and solid angles in the second which are not common to it and the first; let e'', f'', s'' be respectively the numbers of edges, faces, and solid angles in the third which are not common to it and the first or second; and so on. Then we have the following results, namely, the first by Art. 294, and the others by Art. 299:

$$s + f = e + 2,$$
$$s' + f' = e' + 1,$$
$$s'' + f'' = e'' + 1,$$
$$\cdots\cdots\cdots\cdots$$

By addition, since $s + s' + s'' + \ldots = \mathsf{S}$, $f + f' + f'' + \ldots = \mathsf{F}$, and $e + e' + e'' + \ldots = \mathsf{E}$, we obtain

$$\mathsf{S} + \mathsf{F} = \mathsf{E} + \mathsf{P} + 1.$$

301. The following references will be useful to those who study the theory of polyhedrons: EULER, *Novi Commentarii Academiæ Petropolitanæ*, Vol. IV, 1758; LEGENDRE, *Géométrie*, Book VI; POINSOT, *Journal de l'École Polytechnique*, Cahier X; CAUCHY, *Journal de l'École Polytechnique*, Cahier XVI; POINSOT and BERTRAND, *Comptes Rendus ... de l'Académie des Sciences*, Vol. XLVI; CATALAN, *Théorèmes et Problèmes de Géométrie Élémentaire;* KIRKMAN, *Philosophical Transactions* for 1856 and subsequent years; LISTING, *Abhandlungen der Königlichen Gesellschaft ... zu Göttingen*, Vol. X. For the connexion of discontinuous group theory with rotations of a regular polyhedron, see KLEIN, *Das Ikosaeder* (translated by G. G. MORRICE); KLEIN, *Höhere Geometrie*, Vol. II; and SCHOENFLIES, *Krystallsysteme und Krystallstructur* (B. G. Teubner, Leipzig, 1891).

MISCELLANEOUS EXAMPLES.

EXAMPLES XVIII.

1. Find the locus of the vertices of all right-angled spherical triangles having the same hypotenuse; and, from the equation obtained, prove that the locus is a circle when the radius of the sphere is infinite.

2. AB is an arc of a great circle on the surface of a sphere, C its middle point: shew that the locus of the point P, such that the angle APC = the angle BPC, consists of two great circles at right angles to one another. Explain this when the triangle becomes plane.

3. On a given arc of a sphere, spherical triangles of equal area are described: shew that the locus of the angular point opposite to the given arc is defined by the equation

$$\tan^{-1}\left\{\frac{\tan(\alpha+\phi)}{\sin\theta}\right\}+\tan^{-1}\left\{\frac{\tan(\alpha-\phi)}{\sin\theta}\right\}$$
$$+\tan^{-1}\left\{\frac{\tan\theta}{\sin(\alpha+\phi)}\right\}+\tan^{-1}\left\{\frac{\tan\theta}{\sin(\alpha-\phi)}\right\}=\beta.$$

where 2α is the length of the given arc, θ the arc of the great circle drawn from any point P in the locus perpendicular to the given arc, ϕ the inclination of the great circle on which θ is measured to the great circle bisecting the given arc at right angles, and β a constant.

4. In any spherical triangle

$$\tan c=\frac{\cos A\cot a+\cos B\cot b}{\cot a\cot b-\cos A\cos B}.$$

5. If θ, ϕ, ψ denote the distances from the corners A, B, C respectively of the point of intersection of arcs bisecting the angles of the spherical triangle ABC, shew that

$$\cos\theta\sin(b-c)+\cos\phi\sin(c-a)+\cos\psi\sin(a-b)=0.$$

6. If A′, B′, C′ be the poles of the sides BC, CA, AB of a spherical triangle ABC, shew that the great circles AA′, BB′, CC′ meet at a point P, such that

$$\cos PA\cos BC=\cos PB\cos CA=\cos PC\cos AB.$$

7. If O be the point of intersection of arcs AD, BE, CF drawn from the angles of a triangle perpendicular to the opposite sides and meeting them at D, E, F respectively, shew that

$$\frac{\tan AD}{\tan OD},\quad \frac{\tan BE}{\tan OE},\quad \frac{\tan CF}{\tan OF}$$

are respectively equal to

$$1+\frac{\cos A}{\cos B\cos C},\quad 1+\frac{\cos B}{\cos C\cos A},\quad 1+\frac{\cos C}{\cos A\cos B}.$$

8. If p, q, r be the arcs of great circles drawn from the angles of a triangle perpendicular to the opposite sides, (α, α'), (β, β'), (γ, γ') the segments into which these arcs are divided, shew that

$$\tan\alpha\tan\alpha' = \tan\beta\tan\beta' = \tan\gamma\tan\gamma' ;$$

and

$$\frac{\cos p}{\cos\alpha\cos\alpha'} = \frac{\cos q}{\cos\beta\cos\beta'} = \frac{\cos r}{\cos\gamma\cos\gamma'}.$$

9. In a spherical triangle if arcs be drawn from the angles to the middle points of the opposite sides, and if α, α' be the two parts of the one which bisects the side a, shew that

$$\frac{\sin\alpha}{\sin\alpha'} = 2\cos\frac{a}{2}.$$

10. The arc of a great circle bisecting the sides AB, AC of a spherical triangle cuts BC produced at Q: shew that

$$\cos AQ\sin\frac{a}{2} = \sin\frac{c-b}{2}\sin\frac{c+b}{2}.$$

11. If ABCD be a spherical quadrilateral, and the opposite sides AB, CD when produced meet at E, and AD, BC meet at F, the ratio of the sines of the arcs drawn from E at right angles to the diagonals of the quadrilateral is the same as the ratio of those from F.

12. If ABCD be a spherical quadrilateral whose sides AB, DC are produced to meet at P, and AD, BC at Q, and whose diagonals AC, BD intersect at R, then

$$\sin AB\sin CD\cos P \sim \sin AD\sin BC\cos Q = \sin AC\sin BD\cos R.$$

13. If A′ be the angle of the chordal triangle which corresponds to the angle A of a spherical triangle, shew that

$$\cos A' = \sin(S-A)\cos\frac{a}{2}.$$

14. If the tangent of the radius of the circle described about a spherical triangle is equal to twice the tangent of the radius of the circle inscribed in the triangle, the triangle is equilateral.

15. The arc AP of a circle of the same radius as the sphere is equal to the greater of two sides of a spherical triangle, and the arc AQ taken in the same direction is equal to the less; the sine PM of AP is divided at E, so that $\frac{EM}{PM}$ = the cosine of the angle included by the two sides,

and EZ is drawn parallel to the tangent to the circle at Q. Shew that the remaining side of the spherical triangle is equal to the arc QPZ.

16. If through any point P within a spherical triangle ABC great circles be drawn from the angular points A, B, C to meet the opposite sides at a, b, c respectively, prove that

$$\frac{\sin \mathrm{P}a \cos \mathrm{PA}}{\sin \mathrm{A}a} + \frac{\sin \mathrm{P}b \cos \mathrm{PB}}{\sin \mathrm{B}b} + \frac{\sin \mathrm{P}c \cos \mathrm{PC}}{\sin \mathrm{C}c} = 1.$$

17. A and B are two places on the Earth's surface on the same side of the equator, A being further from the equator than B. If the bearing of A from B be more nearly due East than it is from any other place in the same latitude as B, find the bearing of B from A.

18. From the result given in example 18 of page 65 infer the possibility of a regular dodecahedron.

19. A and B are fixed points on the surface of a sphere, and P is any point on the surface. If a and b are given constants, shew that a fixed point S can always be found, in AB or AB produced, such that

$$a \cos \mathrm{AP} + b \cos \mathrm{BP} = s \cos \mathrm{SP},$$

where s is a constant.

20. A, B, C, .. are fixed points on the surface of a sphere; $a, b, c, \ldots$ are given constants. If P be a point on the surface of the sphere, such that

$$a \cos \mathrm{AP} + b \cos \mathrm{BP} + c \cos \mathrm{CP} + \ldots = \text{constant},$$

shew that the locus of P is a circle.

EXAMPLES XIX.

1. One side of a regular spherical quadrilateral $= \cos^{-1}(\frac{1}{3})$, find one of its angles. (R. U. I., 1899.)

2. Prove that the second part of Euclid I, 21 does not necessarily hold on the sphere. Shew where Euclid's proof would fail on the sphere.

Of all the spherical triangles that stand on a given base (supposed less than a quadrant), and whose vertices lie on a given great circle that cuts the base perpendicularly at an internal point of it, find which one has the smallest vertical angle. (R. U. I., 1899.)

3. Prove that in a spherical triangle

$$\frac{\cot^2 \frac{1}{2}\mathrm{B} + \cot^2 \frac{1}{2}\mathrm{C} + 2 \cot \frac{1}{2}\mathrm{B} \cot \frac{1}{2}\mathrm{C} \cos a}{\sin^2 a}$$

is equal to the expression got by replacing B, C, and a by another pair

of angles and the side between them. Find the value of the expression symmetrically in terms of the sides. (R. U. I., 1895.)

4. In a tetrahedron, if a_1, a_4 are a pair of opposite edges, and A_1, A_4 are the dihedral angles at these edges, then the expression

$$\frac{a_1 a_4}{\sin A_1 \sin A_4}$$

has the same value whatever pair be taken. (R. U. I., 1893.)

5. A, B, C are three points on a variable parallel of latitude, the pole being P; the arcs AB and BC are equal; join PB by the arc of a great circle, and let the great circle through A and C cut PB in D; the difference between the longitudes of A and B is constant and is denoted by λ. Shew that DB is greatest when the tangent of the latitude of B equals $\sqrt{\cos \lambda}$. (Sci. and Art, 1895.)

6. A spherical quadrilateral ABCD, whose sides taken in order are a, b, a, b, is inscribed in a small circle. Shew that

$$\tan^2 \tfrac{1}{2} A = \cos \tfrac{1}{2}(a-b) \sec \tfrac{1}{2}(a+b).$$

(Sci. and Art, 1897.)

7. M is the mid point of the side AC of the triangle ABC, and BC is produced to meet in D the great circle through B and M.

If the angles ACD and ABC are equal, shew that $b+c=\pi$.

If the angles ACD and BAC are equal, shew that $BM=\frac{1}{2}\pi$.

(Sci. and Art, 1897.)

8. If the sum of two sides of a spherical triangle is π shew that the sum of the opposite angles is also π.

PAB is a spherical triangle, of which the side AB is fixed, and the angles PAB, PBA are supplementary. Prove that the vertex P lies on a fixed great circle. (Sci. and Art, 1899.)

9. If A, B, C and A′, B′, C′ be the vertices, taken in the same sense in each case, of two trirectangular triangles on a sphere, and if AA′, BB′, CC′ be joined, each of these arcs will be intersected by the other two in points at equal distances from its two extremities.

10. If the opposite sides of a spherical quadrangle are perpendicular to one another, the diagonals are also perpendicular to one another.

If two diagonals of a complete spherical quadrilateral are quadrants, the third also is a quadrant. (JOACHIMSTHAL.)

11. Δ, Δ' are the areas of two faces of a tetrahedron, α the angle at which they intersect, l the length of the edge, and V the volume; shew that

$$2\Delta\Delta' \sin \alpha = 3lV.$$

12. The sides of a spherical quadrilateral inscribed in a small circle are a, b, c, and d; the semi-perimeter is denoted by s, the spherical excess by E, while N and D are defined by the relations

$$N = \sqrt{\{\sin \tfrac{1}{2}(s-a) \sin \tfrac{1}{2}(s-b) \sin \tfrac{1}{2}(s-c) \sin \tfrac{1}{2}(s-d)\}},$$

$$D = \sqrt{\{\cos \tfrac{1}{4}(a+b+c+d) \cos \tfrac{1}{4}(a+b-c-d) \cos \tfrac{1}{4}(a+c-b-d) \cos \tfrac{1}{4}(a+d-b-c)\}}.$$

Prove that

$$\sin \tfrac{1}{4}E = N/\sqrt{\cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c \cos \tfrac{1}{2}d},$$

$$\cos \tfrac{1}{4}E = D/\sqrt{\cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c \cos \tfrac{1}{2}d}.$$ (HART.)

13. Prove that the Jacobian of the angles of a spherical triangle taken with respect to the sides is numerically equal to $\sin A/\sin a$.

(Tripos, 1901.)

14. Definition. A four sided spherical figure ABCD, whose diagonals AC, BD bisect one another, is called a *spherical parallelogram.**

The following properties may easily be proved.

(1) The intersection of diagonals, S, is equidistant from a pair of opposite sides.

S is called the spherical centre of the parallelogram.

(2) If two consecutive angles of a spherical parallelogram are equal, it is inscribable in a small circle.

(3) If two consecutive sides are equal the parallelogram is circumscribable to a small circle.

(4) The figure reciprocal to a spherical parallelogram is another parallelogram having the same spherical centre.

(5) The intersections of a pair of opposite sides lie on the polar great circle of S.

(6) Two consecutive corners of a parallelogram and the antipodes of the other two corners lie on a small circle.

(7) If A, B be fixed points on a given small circle, and P, Q variable points on the antipodal small circle, such that PQ = AB, then the figure formed by the great circles AP, PQ, QB, BA is a parallelogram.

(8) The area of this parallelogram is the same wherever P and Q may be on the small circle, provided PQ = AB. Its spherical excess is four times the angle between the great and small circles AB.

(9) The area of the triangle ABP is constant.

(10) The sum of the angles of the triangle formed by the great arcs PA, PB and the small-circle arc AB is two right angles.

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* EULER, *Nov. Act. Petrop.*, X, p. 57. GUDERMANN, *Niedere Sphärik*, §§ 78-96. BALTZER, *Stereometrie*, § IV, 16.

CHAPTER XIX.

THE EXTENDED DEFINITION OF THE SPHERICAL TRIANGLE.

302. In the foregoing chapters it has been convenient to consider only those spherical triangles which comply with the conventional restrictions of Articles 22 and 23, namely that none of their sides or angles shall exceed two right angles. For the practical applications of Spherical Trigonometry such triangles are the only ones with whose properties it is necessary to be familiar; but the subject does not take its proper place in the science of Pure Mathematics unless its full generality is preserved by the discarding of a restriction which, viewed from a theoretical standpoint, is arbitrary and unnecessary. MÖBIUS * (in 1846) was the first to extend the traditional definition of the spherical triangle and to shew that the parts of the generalised triangle satisfy the same fundamental formulae as those of the original. It is proposed in the present chapter to consider very briefly the ideas underlying this important generalisation, and to obtain a few of the most fundamental results: for a complete account of

* "*Ueber eine neue Behandlungsweise der analytischen Sphärik*" (1846), *Ges. Werke, Bd. II*, and "*Entwickelung der Grundformeln der sphärischen Trigonometrie in grösstmöglicher Allgemeinheit*" (*Verhandlungen der Kön. Säch. Gesellschaft der Wiss. zu Leipzig*, 1860). See also CHAUVENET, *Trigonometry*, Part II, Chap. IV. CHAUVENET does not assign directions to the great circles which form the triangle. Cp. Art. 39.

the subject the reader is referred to Dr. E. STUDY'S * memoir on *Spherical Trigonometry and Orthogonal Substitutions*, from which the substance of several of the following Articles is taken.

303. To begin with, since a complete turn, that is a revolution through an angle of 2π, leaves unchanged the position of the object turned, it is natural and convenient to regard angles or arcs of great circles, which differ from one another by multiples of 2π, as equal to one another. This convention justifies in all respects (save one, to be considered in due course), the assumption with which we shall set out, that every side and angle of any spherical triangle lies between the limits 0 and 2π. On this understanding, the number of triangles having three given points on the sphere for corners, though greater than unity, is not infinite.

304. Further, in considering the angles about any point on the surface of the sphere, we must select a *sense*, or direction of turning, in which sense the angles are to be reckoned positively; say the sense contrary to that of the rotation of the hands of a watch laid face upwards on the outside of the surface of the sphere. If we imagine the watch to be moved over the whole surface of the sphere, we get for every point on the surface a definite sense in which rotations are to be measured positively.

305. Definition of the spherical triangle. Now let A, B, C be three points on the sphere, and let them be joined by great circles. Along each of these great circles let us choose arbitrarily a positive direction; and let us denote by a, b, c the arcs, whose magnitudes are contained between 0 and 2π,

* E. STUDY, *Sphärische Trigonometrie, Orthogonale Substitutionen und Elliptische Functionen.* (*Abhl. der Kön. Säch. Gesellschaft der Wiss. zu Leipzig, Bd. XX*).

which are traversed by a point starting from B and moving in the positive direction along the great circle BC to C, then in the positive direction along CA to A, and then in the positive direction along AB to B. Further, we denote by α the angle, whose magnitude is contained between 0 and 2π, through which the great circle CA must be turned in the positive sense about A in order to bring its positive direction into coincidence with the positive direction of the great circle AB; β and γ are defined in a similar manner.

The six elements a, b, c, α, β, γ, thus defined, may be said to constitute the spherical triangle ABC. The angles α, β, γ are called the *angles* of the triangle, the arcs a, b, c the *sides*.

306. It should be noticed that this triangle, unlike the old traditional spherical triangle, is not sufficiently defined by the positions of its three corners. The complete specification involves also (1st) an arbitrary sense for rotations about points on the sphere, (2nd) positive directions arbitrarily assigned to the three great circles, (3rd) the order in which the corners occur in the name of the triangle (*i.e.*, other things being unchanged, the triangle ABC has not the same elements as the triangle ACB).

307. It must also be carefully noted that the angles α, β, γ of the generalised spherical triangle as here defined correspond, in the particular case of a triangle whose sides are all less than π, not to the *interior* angles which in former chapters we have denoted by A, B, C, but to their *supplements.* The substitution of Greek for Roman letters in naming the angles of the generalised triangle will be a sufficient safeguard from confusion between the former and the present meaning attached to the phrase "angles of the spherical triangle."

308. The polar triangle. Every great circle has two poles,

but when one of the two directions in which a great circle may be traversed by a point has been arbitrarily assigned to it as its positive direction, one of the two poles is at the same time, by a certain convention, associated with it as its pole, and the ambiguity thus removed. We choose, in fact, that one of the two poles which would lie to the left of a person walking on the outside of the sphere and traversing the great circle in the positive direction.

The polar triangle of a given spherical triangle is then defined to be the triangle whose corners are the poles of the sides of the original triangle, and whose sides have for poles the corners of the original triangle. The relation between the two triangles is completely reciprocal, and it is readily seen that the sides of either are equal respectively to the corresponding angles of the other. So that spherical triangles occur in pairs, the members of which are derived from one another by interchange of angles and sides.

309. Different triangles having the same corners. In considering the number of triangles that can be made having three given points for corners, we notice that we have, in the case of each of the three great circles joining these points, a choice of two directions, either of which may be taken as the positive one. Consequently there are $2 \times 2 \times 2$ or 8 different possible triangles. And if we further reserve the right to change the arbitrarily assigned positive sense for angles about a point on the sphere, the number of triangles having given corners becomes 16. The following diagrams shew (in stereographic projection) the forms of four of the first named eight triangles, corresponding to essentially different types. The others can be easily derived from them. The sides are printed more heavily than the remaining arcs of the great circles of which they form part, and the angles are indicated by light arrowed curves.
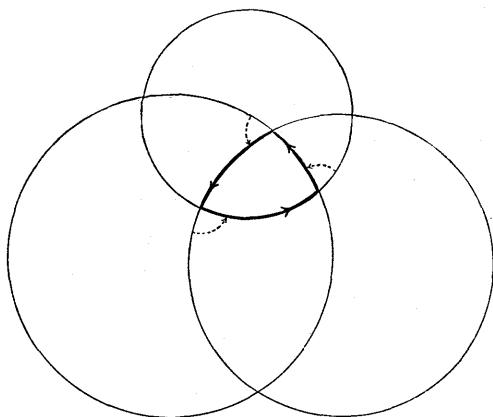
Fig. 1.
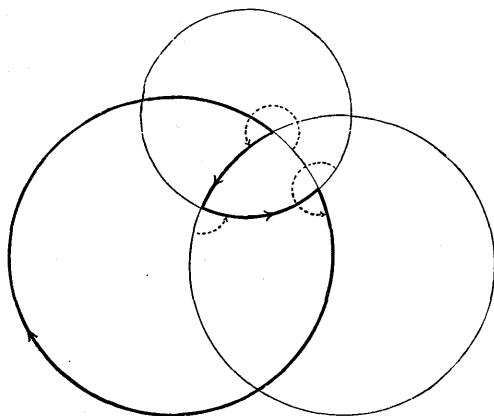
Fig. 2.
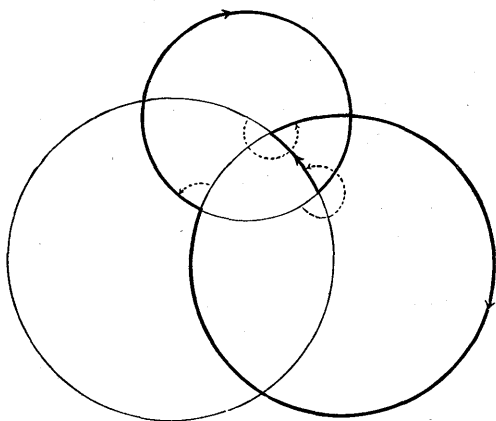
FIG. 3.
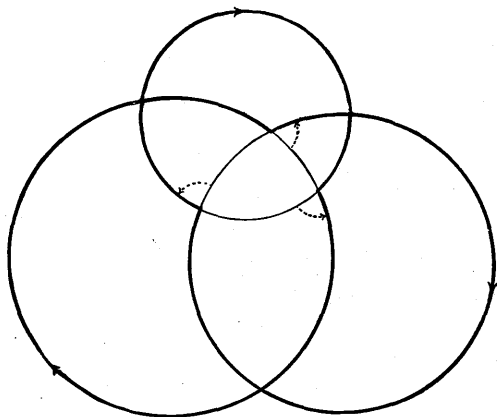
FIG. 4.

310. It will be convenient to distinguish the elements of the eight triangles having the same corners by the numbers 1, 2, ..., 8, used as suffixes, and we may assign the suffix 1 to the particular triangle whose sides are less than semicircles. If the suffix 2 be given to that which is derived from the first by reversing the direction belonging to the great circle BC, it is seen, by comparison of Figures 1 and 2, that

$$\left.\begin{array}{lll} a_2 = 2\pi - a_1, & b_2 = b_1, & c_2 = c_1, \\ \alpha_2 = \alpha_1, & \beta_2 = \pi + \beta_1, & \gamma_2 = \pi + \gamma_1. \end{array}\right\} \quad \ldots\ldots\ldots\ldots(\text{i})$$

The corresponding substitutions for the other triangles are obtained equally readily, and the complete set of results is represented in the following table:

TABLE I.

	a	b	c	α	β	γ
(1)	a_1	b_1	c_1	α_1	β_1	γ_1
(2)	$2\pi - a_1$	b_1	c_1	α_1	$\pi + \beta_1$	$\pi + \gamma_1$
(3)	a_1	$2\pi - b_1$	c_1	$\pi + \alpha_1$	β_1	$\pi + \gamma_1$
(4)	a_1	b_1	$2\pi - c_1$	$\pi + \alpha_1$	$\pi + \beta_1$	γ_1
(5)	a_1	$2\pi - b_1$	$2\pi - c_1$	α_1	$\pi + \beta_1$	$\pi + \gamma_1$
(6)	$2\pi - a_1$	b_1	$2\pi - c_1$	$\pi + \alpha_1$	β_1	$\pi + \gamma_1$
(7)	$2\pi - a_1$	$2\pi - b_1$	c_1	$\pi + \alpha_1$	$\pi + \beta_1$	γ_1
(8)	$2\pi - a_1$	$2\pi - b_1$	$2\pi - c_1$	α_1	β_1	γ_1

Only one type of transformation, namely that corresponding to the reversal of the direction assigned to a great circle, is necessary for the derivation of any seven of these triangles from the eighth. By it, for instance, (2), (3), and (4) are derived from (1); (5), (6), and (7) from (2), (3), and (4); and (8) from (5), (6), or (7). It will be noticed that the first diagram represents (1); the second (2), (3), or (4); the third (5), (6), or (7); and the fourth (8).

311. Inequalities satisfied by sides and angles. When the three sides of a spherical triangle, of the kind considered in previous chapters, are given, the angles are determinate; but given arcs a, b, c can be combined to form a real triangle only if they satisfy the inequalities

$$a+b+c \leq 2\pi, \quad b+c-a \geq 0, \quad c+a-b \geq 0, \quad a+b-c \geq 0.$$

In like manner a triangle having given angles A, B, C, is possible only if

$$\mathrm{A}+\mathrm{B}+\mathrm{C} \geq \pi,$$

$$\pi-\mathrm{B}-\mathrm{C}+\mathrm{A} \geq 0, \quad \pi-\mathrm{C}-\mathrm{A}+\mathrm{B} \geq 0, \quad \pi-\mathrm{A}-\mathrm{B}+\mathrm{C} \geq 0.$$

Let us now introduce, for the generalised triangle, the notation:

$$\begin{aligned} 2s &= 2\pi-a-b-c, & 2\sigma &= 2\pi-\alpha-\beta-\gamma, \\ 2s' &= -a+b+c, & 2\sigma' &= -\alpha+\beta+\gamma, \\ 2s'' &= a-b+c, & 2\sigma'' &= \alpha-\beta+\gamma, \\ 2s''' &= a+b-c, & 2\sigma''' &= \alpha+\beta-\gamma, \end{aligned}$$

and it is seen at once that the corresponding inequalities, which must be satisfied by the sides and angles of the eight sorts of spherical triangle respectively, are as indicated in the following tables:

TABLE II.

	s	s'	s''	s'''
(1)	≥ 0	≥ 0	≥ 0	≥ 0
(2)	≤ 0	≤ 0	$\leq \pi$	$\leq \pi$
(3)	≤ 0	$\leq \pi$	≤ 0	$\leq \pi$
(4)	≤ 0	$\leq \pi$	$\leq \pi$	≤ 0
(5)	$\geq -\pi$	$\geq \pi$	≥ 0	≥ 0
(6)	$\geq -\pi$	≥ 0	$\geq \pi$	≥ 0
(7)	$\geq -\pi$	≥ 0	≥ 0	$\geq \pi$
(8)	$\leq -\pi$	$\leq \pi$	$\leq \pi$	$\leq \pi$

TABLE III.

	σ	σ'	σ''	σ'''
(1)	≥ 0	≥ 0	≥ 0	≥ 0
(2)	$\geq -\pi$	$\geq \pi$	≥ 0	≥ 0
(3)	$\geq -\pi$	≥ 0	$\geq \pi$	≥ 0
(4)	$\geq -\pi$	≥ 0	≥ 0	$\geq \pi$
(5)	$\geq -\pi$	$\geq \pi$	≥ 0	≥ 0
(6)	$\geq -\pi$	≥ 0	$\geq \pi$	≥ 0
(7)	$\geq -\pi$	≥ 0	_ 0	$\geq \pi$
(8)	≥ 0	≥ 0	≥ 0	≥ 0

312. So far we have considered only eight of the possible triangles having given corners. The other eight are got by reversing the sense in which angles at any point on the sphere are to be measured positively, which is of course equivalent to taking $2\pi - \alpha$, $2\pi - \beta$, $2\pi - \gamma$ instead of α, β, γ respectively. Hence we have the following relations, wherein accented suffixes indicate association with the second set of eight triangles:

$$\left.\begin{array}{ll} \sigma_{r'} = -\pi - \sigma_r, & \sigma'_{r'} = \pi - \sigma'_r, \\ \sigma''_{r'} = \pi - \sigma''_r, & \sigma'''_{r'} = \pi - \sigma'''_r, \end{array}\right\} (r = 1, 2, \ldots 8).$$

Thus, while the inequalities for the sides of the second set of triangles are the same as those for the first set, the inequalities for the angles are different, being as shewn in Table IV.

TABLE IV.

	σ	σ'	σ''	σ'''
(1′)	$\leq -\pi$	$\leq \pi$	$\leq \pi$	$\leq \pi$
(2′)	≤ 0	≤ 0	$\leq \pi$	$\leq \pi$
(3′)	≤ 0	$\leq \pi$	≤ 0	$\leq \pi$
(4′)	≤ 0	$\leq \pi$	$\leq \pi$	≤ 0
(5′)	≤ 0	≤ 0	$\leq \pi$	$\leq \pi$
(6′)	≤ 0	$\leq \pi$	≤ 0	$\leq \pi$
(7′)	≤ 0	$\leq \pi$	$\leq \pi$	≤ 0
(8′)	$\leq -\pi$	$\leq \pi$	$\leq \pi$	$\leq \pi$

313. With regard to the polar triangle of any one of these triangles, we can find to which type it belongs from the consideration that the angles and sides of the polar triangle will satisfy the same sets of inequalities as the sides and angles of the original triangle respectively.

If, for example, we take the triangle (3), the angles of its polar triangle must satisfy inequalities corresponding to the 3rd line in Table II. Looking for these inequalities in the angle tables, we find them only in the 3rd and 6th lines of Table IV. Hence the polar triangle is either of type (3′) or of type (6′). Again, as the angles of (3) satisfy the 3rd line of inequalities in Table III, the sides of its polar triangle satisfy the same inequalities, and these we find only in the 6th line of Table II, which belongs to types (6) and (6′). Thus it appears that the polar triangle of a triangle of type (3) is a triangle of type (6′).

314. The fundamental formulae. The *cosine formulae,*

proved in Chapter III for the restricted spherical triangle, assume the following form, when the new notation is adopted:

$$\left.\begin{aligned}\cos a &= \cos b \cos c - \sin b \sin c \cos \alpha,\\ \cos b &= \cos c \cos a - \sin c \sin a \cos \beta,\\ \cos c &= \cos a \cos b - \sin a \sin b \cos \gamma,\end{aligned}\right\} \ldots\ldots\ \ldots\ldots\text{(ii)}$$

Suppose now that these relations are satisfied by six quantities $a, b, c, \alpha, \beta, \gamma$, and that we make the substitutions,

$$\left.\begin{matrix} a = 2\pi - a', & b = b', & c = c', \\ \alpha = \alpha' & \beta = \beta' - \pi, & \gamma = \gamma' - \pi, \end{matrix}\right\} \ldots\ldots\ldots\ldots\text{(iii)}$$

we find that the quantities represented by the accented letters satisfy relations of precisely the same form. That is to say the formulae (ii) are unaltered by the substitution (iii).

Now we know that the formulae (ii) are true for a triangle of type (1); and we have seen that the elements of the triangles (2), (3), (4), are derived from those of (1) by the substitution (iii) and the two others symmetrical with it. Hence it appears that the formulae (ii) are true for the triangles (2), (3), (4).

And since from the triangles (2), (3), (4), we derive (5), (6), (7), by the same substitutions, and (8) likewise from one of the latter set of three, it follows that the formulae (ii) are valid for the eight types (1), (2),... (8).

Further, the formulae are unaltered when we put $2\pi - \alpha$ $2\pi - \beta$, $2\pi - \gamma$, instead of α, β, γ respectively; hence they are valid also for the types (1′), (2′),... (8′).

Thus the cosine formulae are seen to be true for the generalised spherical triangle.

315. The supplementary set of *cosine formulae*, namely

$$\left.\begin{aligned}\cos \alpha &= \cos \beta \cos \gamma - \sin \beta \sin \gamma \cos a,\\ \cos \beta &= \cos \gamma \cos \alpha - \sin \gamma \sin \alpha \cos b,\\ \cos \gamma &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \cos c,\end{aligned}\right\} \ldots\ldots\ \ldots\ldots\text{(iv)}$$

may be treated in the same manner. They are unaltered by

the substitution (iii), and therefore, having been established for triangles of type (1), are equally true for the sixteen types of the generalised triangle.

316. The same reasoning applies to the *sine formulae*

$$\frac{\sin a}{\sin \alpha}=\frac{\sin b}{\sin \beta}=\frac{\sin c}{\sin \gamma}, \quad \text{(v)}$$

establishing their generality, and therefore that also of the formulae derived from them, namely

$$\left.\begin{aligned} 2n &= \sin b \sin c \sin \alpha = \sin c \sin a \sin \beta = \sin a \sin b \sin \gamma, \\ 2\mathsf{N} &= \sin \beta \sin \gamma \sin a = \sin \gamma \sin \alpha \sin b = \sin \alpha \sin \beta \sin c, \end{aligned}\right\} \text{(vi)}$$

and

$$\left.\begin{aligned} 4n^2 &= 4 \sin s \sin s' \sin s'' \sin s''' \\ &= 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c, \\ 4\mathsf{N}^2 &= 4 \sin \sigma \sin \sigma' \sin \sigma'' \sin \sigma''' \\ &= 1 - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma + 2 \cos \alpha \cos \beta \cos \gamma. \end{aligned}\right\} \text{(vii)}$$

317. The *cotangent formulae* of Art. 49, and the formulae of the type

$$\left.\begin{aligned} \sin a \cos b + \sin c \cos \beta + \cos a \sin b \cos \gamma = 0, \\ \sin a \cos c + \sin b \cos \gamma + \cos a \sin c \cos \beta = 0, \end{aligned}\right\} \quad \text{(viii)}$$

(which are those discussed in Art. 53,) are also of general application, being deductions from the sine and cosine formulae.

318. Thus all the *fundamental* formulae of the restricted triangle are likewise formulae of the sixteen types of the generalised triangle.

319. Delambre's analogies. In generalising those formulae which, either in their final form or in the course of their deduction from the fundamental formulae, involve the taking of a square root, we have to proceed with care in the matter of the ambiguous sign. For it will be remembered that, in the case of the restricted triangle, the choice of sign was determined from consideration of the limitations which had

been arbitrarily imposed on the elements. (See, for example, Art. 56.) Now the restrictions upon the values of the elements being different for the 16 types of the generalised triangle, a diversity of signs is to be expected.

320. From the formulae (ii), proved valid for the generalised triangle, the following are immediate inferences.

$$\left.\begin{aligned} \sin^2\tfrac{1}{2}\alpha &= \frac{\sin s \sin s'}{\sin b \sin c}, & \cos^2\tfrac{1}{2}\alpha &= \frac{\sin s'' \sin s'''}{\sin b \sin c}, \\ \sin^2\tfrac{1}{2}\beta &= \frac{\sin s \sin s''}{\sin c \sin a}, & \cos^2\tfrac{1}{2}\beta &= \frac{\sin s''' \sin s'}{\sin c \sin a}, \\ \sin^2\tfrac{1}{2}\gamma &= \frac{\sin s \sin s'''}{\sin a \sin b}, & \cos^2\tfrac{1}{2}\gamma &= \frac{\sin s' \sin s''}{\sin a \sin b}; \end{aligned}\right\} \ldots\ldots(\text{ix})$$

and from (ix) again we derive

$$\frac{\sin\frac{1}{2}\beta\cos\frac{1}{2}\gamma}{\sin\frac{1}{2}\alpha} = \sqrt{\frac{\sin^2 s''}{\sin^2 a}}, \quad \frac{\cos\frac{1}{2}\beta\sin\frac{1}{2}\gamma}{\sin\frac{1}{2}\alpha} = \sqrt{\frac{\sin^2 s'''}{\sin^2 a}}. \ \ldots.(\text{x})$$

Now in these two expressions the particular square roots to be chosen are not independent of one another. In fact if the former root be

$$\epsilon\,.\frac{\sin s''}{\sin a}, \quad (\epsilon = \pm 1),$$

the latter must at the same time be

$$\epsilon\,.\frac{\sin s'''}{\sin a}.$$

For, if we denote the latter root by

$$\epsilon'.\frac{\sin s'''}{\sin a}, \quad (\epsilon' = \pm 1),$$

then we get, on multiplication,

$$\frac{\sin\beta\sin\gamma}{4\sin^2\frac{1}{2}\alpha} = \epsilon\epsilon'.\frac{\sin s''\sin s'''}{\sin^2 a};$$

which, in virtue of formulae (ix) and (vii), reduces to

$$\sin^2 a \sin b \sin c \sin\beta \sin\gamma = \epsilon\epsilon'\,.\,4n^2,$$

so that $\epsilon\epsilon' = +1$. Thus $\epsilon = \epsilon'$.

By adding and subtracting the formulae (x) we get the first two of DELAMBRE'S analogies:

$$\left.\begin{aligned}\frac{\sin\frac{1}{2}(\beta+\gamma)}{\sin\frac{1}{2}\alpha}&=\pm\frac{\cos\frac{1}{2}(b-c)}{\cos\frac{1}{2}a}, & \frac{\sin\frac{1}{2}(\beta-\gamma)}{\sin\frac{1}{2}\alpha}&=\mp\frac{\sin\frac{1}{2}(b-c)}{\sin\frac{1}{2}a}\\ \frac{\cos\frac{1}{2}(\beta+\gamma)}{\cos\frac{1}{2}\alpha}&=\mp\frac{\cos\frac{1}{2}(b+c)}{\cos\frac{1}{2}a}, & \frac{\cos\frac{1}{2}(\beta-\gamma)}{\cos\frac{1}{2}\alpha}&=\pm\frac{\sin\frac{1}{2}(b+c)}{\sin\frac{1}{2}a}\end{aligned}\right\}\text{(A) (xi)}$$

321. From the mode of proof it is seen that in the first row the upper signs are to be taken together, and the lower signs together. But further, throughout the whole group (A) of formulae the upper signs go together, and the lower signs together. There are several ways of seeing this; we may, for example, derive the two left-hand formulae simultaneously by a method similar to that adopted in deriving the two formulae of the upper row. Or we may infer it by comparing the upper formulae with the factorised form of the equalities

$$\frac{\sin\beta\pm\sin\gamma}{\sin\alpha}=\frac{\sin b\pm\sin c}{\sin a},$$

which are immediate inferences from the fundamental *sine formulae.*

322. In addition to the formulae (A) there are two other groups of formulae, which we may denote by (B) and (C), derived from (A) by cyclic interchange of the letters a, b, c, and of α, β, γ. In each of these groups, taken by itself, all the upper signs go together, and all the lower signs go together.

323. It can be shewn further that when the three groups of formulae (A), (B), (C) are considered *together*, we must take either *only the upper signs*, or *only the lower signs.*

For DELAMBRE'S analogies are relations between four quantities, of the form
$$\frac{w}{x}=\pm\frac{y}{z}.$$

If we put these in the form

$$\frac{w-x}{w+x}=\frac{y\mp z}{y\pm z},$$

then from the formulae (A) we get, when we take the upper signs,

$$\left.\begin{array}{l}\tan\frac{\sigma}{2}\tan\frac{\sigma'}{2}=\tan\frac{s''}{2}\tan\frac{s'''}{2},\ \ \tan\frac{\sigma''}{2}\cot\frac{\sigma'''}{2}=\cot\frac{s''}{2}\tan\frac{s'''}{2},\\ \tan\frac{\sigma}{2}\cot\frac{\sigma'}{2}=\cot\frac{s}{2}\tan\frac{s'}{2},\ \ \tan\frac{\sigma''}{2}\tan\frac{\sigma'''}{2}=\tan\frac{s}{2}\tan\frac{s'}{2};\end{array}\right\}\text{(A) (xii)}$$

and, when we take the lower signs,

$$\left.\begin{array}{l}\tan\frac{\sigma}{2}\tan\frac{\sigma'}{2}=\cot\frac{s''}{2}\cot\frac{s'''}{2},\ \ \tan\frac{\sigma''}{2}\cot\frac{\sigma'''}{2}=\tan\frac{s''}{2}\cot\frac{s'''}{2},\\ \tan\frac{\sigma}{2}\cot\frac{\sigma'}{2}=\tan\frac{s}{2}\cot\frac{s'}{2},\ \ \tan\frac{\sigma''}{2}\tan\frac{\sigma'''}{2}=\cot\frac{s}{2}\cot\frac{s'}{2}.\end{array}\right\}\text{(A) (xiii)}$$

If now the corresponding formulae (B) and (C) be formed, it will be seen at once that the assumption, that in any one of the groups (A), (B), (C) the upper sign can be taken, while at the same time the lower sign is taken in another, is inadmissible.

324. Proper and improper triangles. Those triangles for which the upper signs must be taken in DELAMBRE'S analogies are called *proper triangles.* Those for which the lower signs must be taken are called *improper triangles.**

325. NAPIER'S formulae for $\tan\frac{1}{2}(b\pm c)$, etc., derived from those of DELAMBRE by division, are the same for proper as for improper triangles.

326. Indeed all the trigonometrical formulae of proper triangles may be divided into two classes, the first those which are valid equally for proper and for improper triangles, the second those which hold good only for proper triangles. The formulae of the first class are founded on the sine and cosine

* This distinction between the two sorts of triangles corresponds to the distinction between *proper* and *improper* orthogonal substitutions, *i.e.* those for which the determinant of the coefficients equals $+1$, and those for which it equals -1, respectively.

formulae, those of the second on DELAMBRE'S analogies. From these root formulae we may pass to all others of the same class by unambiguous operations, and so likewise we may pass from formulae of the second class to formulae of the first. But the transition from formulae of the first class to formulae of the second requires the determination of the sign of a square root, that is to say a choice between two alternatives.

327. It is easy to see that the upper signs in DELAMBRE'S analogies are those which must be taken for a triangle whose sides and angles are less than π. And indeed it will be found that the triangles of types (1), (2), (3), (4) are proper triangles, while (5), (6), (7), (8) are improper.

328. Further, if we increase a single side or a single angle of a triangle by 2π, we thereby change the signs in DELAMBRE'S analogies. This circumstance would be of importance if we were to extend our discussion to triangles whose sides and angles are not restricted to be less than 2π. It emphasises the obvious fact that, while angles which differ by multiples of 2π may be regarded as identical so long as we use only their own trigonometrical functions, they can not be so regarded in expressions where the trigonometrical functions of their sub-multiples are employed.

329. It might be supposed, since the substitution (iii), applied to any one of DELAMBRE'S analogies, does not alter the sign, and since all the triangles (2), (3) ... (8) are derived from (1) by one or more substitutions of this type, that all the eight triangles are proper triangles. But, as a matter of fact, when the substitution (i) is applied to the triangle (4), for example, (see Table I), we do not *immediately* get the triangle (6). The set of elements that results is

$$\left.\begin{array}{lll} a=2\pi-a_1, & b=b_1, & c=2\pi-c_1 \\ \alpha=\pi+\alpha_1, & \beta=2\pi+\beta_1, & \gamma=\pi+\gamma_1, \end{array}\right\} \quad \ldots\ldots\text{(xiv)}$$

corresponding to a triangle whose angle β is greater than 2π.

We have to subtract 2π from β in order to bring the triangle within the stipulated restriction, and then we arrive at type (6). It is this subtraction of 2π which, though leaving the fundamental formulae unchanged, alters the signs in DELAMBRE'S analogies, and brings type (6) into the class of improper triangles.

330. Another proof of Delambre's analogies.*

From the sine formulae we have

$$\sin\beta\sin a = \sin b\sin\alpha,$$
$$\sin\gamma\sin a = \sin c\sin\alpha;$$

whence

$$(\sin\beta - \sin\gamma)\sin a = (\sin b - \sin c)\sin\alpha, \quad\ldots\ldots\ldots(\text{xv})$$
$$(\sin\beta + \sin\gamma)\sin a = (\sin b + \sin c)\sin\alpha.\quad\ldots\ldots\ldots(\text{xvi})$$

Again from (viii)

$$\sin b\cos c + \sin a\cos\gamma + \cos b\sin c\cos\alpha = 0,$$
$$\sin c\cos b + \sin a\cos\beta + \cos c\sin b\cos\alpha = 0;$$

whence by addition

$$(\cos\beta + \cos\gamma)\sin a + \sin(b+c).(1+\cos\alpha) = 0. \quad\ldots(\text{xvii})$$

The correlative formula,

$$\sin(\beta+\gamma).(1+\cos a) + (\cos b + \cos c)\sin\alpha = 0 \quad\ldots(\text{xviii})$$

is deduced by consideration of the polar triangle.

If, now, in equations (xv)...(xviii) we factorise all the terms, and introduce the abbreviations

$$\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}a = l, \qquad \sin\tfrac{1}{2}(b-c)\sin\tfrac{1}{2}\alpha = \lambda,$$
$$\cos\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}a = m, \qquad \cos\tfrac{1}{2}(b-c)\sin\tfrac{1}{2}\alpha = \mu,$$
$$\sin\tfrac{1}{2}(\beta+\gamma)\cos\tfrac{1}{2}a = n, \qquad \sin\tfrac{1}{2}(b+c)\cos\tfrac{1}{2}\alpha = \nu,$$
$$\cos\tfrac{1}{2}(\beta+\gamma)\cos\tfrac{1}{2}a = p, \qquad \cos\tfrac{1}{2}(b+c)\cos\tfrac{1}{2}\alpha = \varpi,$$

we get

$$\left.\begin{array}{ll} lp = \lambda\varpi, & mn = \mu\nu, \\ mp = -\nu\varpi, & np = -\mu\varpi. \end{array}\right\}\ldots\ldots\ldots\ldots(\text{xix})$$

* CHAUVENET, *Spherical Trigonometry*, §§ 25-27; BALTZER, *Trigonometrie*, § 5, X.

Now multiply the last two equations and divide by the second, and there results

$$p^2 = \varpi^2.$$

Hence either (1°) $p = \varpi,$

and at the same time $l = \lambda,\quad m = -\nu,\quad n = -\mu;$

or (2°) $p = -\varpi,$

and at the same time $l = -\lambda,\quad m = \nu,\quad n = \mu.$

These relations are the group (A) of DELAMBRE'S formulae.

331. L'Huilier's Formulae. These are formulae of the class that are not the same for proper as for improper triangles. For proper triangles they are deduced from the equalities (xii), and are simplified in form by the introduction of a new symbol. So we have the following relations, of which the first is to be regarded as the definition of L:

$$\left.\begin{aligned}\tan\frac{\sigma}{2}\tan\frac{\sigma'}{2}\tan\frac{\sigma''}{2}\tan\frac{\sigma'''}{2} &= \mathsf{L}^2\\ &= \tan\frac{s}{2}\tan\frac{s'}{2}\tan\frac{s''}{2}\tan\frac{s'''}{2},\end{aligned}\right\}\quad\text{......(xx)}$$

$$\left.\begin{aligned}\tan\frac{s}{2}\tan\frac{\sigma}{2}\tan\frac{s'}{2}\tan\frac{\sigma'}{2} &= \mathsf{L}^2,\\ \tan\frac{s''}{2}\tan\frac{\sigma''}{2}\tan\frac{s'''}{2}\tan\frac{\sigma'''}{2} &= \mathsf{L}^2,\end{aligned}\right\}\quad\text{............(xxi)}$$

and four others got by cyclic interchange of singly, doubly, and triply accented letters,

$$\left.\begin{aligned}\tan\frac{s}{2}\tan\frac{\sigma}{2} = \tan\frac{s'}{2}\tan\frac{\sigma'}{2} &= \tan\frac{s''}{2}\tan\frac{\sigma''}{2}\\ = \tan\frac{s'''}{2}\tan\frac{\sigma'''}{2} &= \mathsf{L}.\end{aligned}\right\}\quad\text{..........(xxii)}$$

The definition of L involves the taking of a square root; for triangles whose sides and angles lie between 0 and π the positive root is to be chosen.

Corresponding results for the improper triangles can be derived from formulae (xiii).

332. The following references will be useful to the reader who is interested in the wider views of Spherical Trigonometry and in its relations with other branches of Pure Mathematics.

E. STUDY. "*Sphärische Trigonometrie, etc.*," *Abhandlungen der Kön. Säch. Gesellschaft der Wiss. zu Leipzig*, XX, 1893.

Mathematical Report to the International Congress at Chicago, 1893.

"*Mathematische Mitteilungen*," *Verhandlungen der Kön. Säch. Gesellschaft der Wiss. zu Leipzig*, 1895, p. 532.

"*Some Researches in Spherical Trigonometry*," *Papers published by The American Mathematical Society*, I, 1896, p. 382.

MRS. G. CHISHOLM YOUNG, *Algebraisch gruppentheoretische Untersuchungen zur sphärischen Trigonometrie*, Göttingen, 1895.

"*Sulla varietà razionale normale* M_3^4 *di* S_6 *rappresentante della trigonometria sferica*," *Atti della R. Accademia delle Scienze di Torino*, XXXIV, 1899, p. 587.

F. KLEIN, *Vorlesungen über die hypergeometrische Function*, Göttingen, 1894, p. 285 *et seq.*

Vorlesungen über die nicht-Euklid'sche Geometrie, Göttingen, 1893, Vol. I, p. 11; Vol. II, p. 169.

F. MEYER, "*Der Resultantenbegriff in der sphärischen Trigonometrie*," *Crelle's Journal*, CXV, p. 209.

"*Die Resultantenbildungen der Trigonometrie*," *Jahresbericht der Deut. Math. Vereinigung*, IV, 1894-5, p. 92.

O. PUND, "*Ueber Substitutionsgruppen in der sphärischen Trigonometrie, inbesondere die Neperschen Regeln für die rechtwinklige sphärischen Dreiecke*," *Mitteilungen der Math. Gesell. in Hamburg*, III, 1897-8, p. 290.

R. F. MUIRHEAD, *Proc. Edinburgh Math. Soc.* 1894-5, p. 129.

STÉPHANOS, "*Sur la relation qui existe entre le problème de la trigonométrie sphérique et la théorie du système de trois formes biquadratiques binaires*," *Bulletin de la Société mathématique de France*, X, 1882, p. 134.

DZIOBEK, "*Ueber eine Erweiterung des Gauss'schen Pentagramma mirificum auf ein beliebiges sphärischen Dreieck*," *Grunert's Archiv*, XIX, XX.

CHAPTER XX.

APPLICATIONS OF DETERMINANTS TO SPHERICAL GEOMETRY.

333. Normal coordinates with respect to a trirectangular triangle. The use of the normal coordinates with respect to a triangle, defined in Art. 162, renders possible an analytical method in spherical geometry analogous to the method of trilinear coordinates in plane geometry.

For this purpose it is convenient to take as triangle of reference a triangle whose three sides are quadrants, and whose three angles are consequently right angles. A triangle of this sort is called a *trirectangular triangle*; its principal properties have been discussed in Chapter XVII. In such a triangle each corner is the pole of the opposite side; and consequently the arc joining a corner to any point in the opposite side is a quadrant. If ABC be the triangle, P any point on the sphere, and D, E, F the points where AP, BP, CP intersect the sides (see figure of Art. 160), AP, BP, CP are complementary to PD, PE, PF respectively. Thus the normal coordinates of P, already defined as sin PD, sin PE, sin PF, are, in the present instance, the same as cos AP, cos BP, cos CP, and will be denoted by λ, μ, ν as in Art. 272. The student of Analytical Solid Geometry will recognise in λ, μ, ν the direction cosines of OP, referred to the rectangular axes OA, OB, OC, where O is the centre of the sphere.

334. Fundamental properties of the coordinates. Two properties of the normal coordinates of a point with respect to

a trirectangular triangle are fundamental, and will be used constantly throughout the present chapter.

(1) *If* λ, μ, ν *be the coordinates of any point* P,

$$\lambda^2 + \mu^2 + \nu^2 = 1. \quad\quad (1)$$

(2) *If* λ, μ, ν *and* λ', μ', ν' *be the coordinates of any two points* P *and* P′,

$$\lambda\lambda' + \mu\mu' + \nu\nu' = \cos PP'. \quad\quad (2)$$

Proofs of these propositions have been given in Articles 269 and 270.

335. Assumption with regard to the triangle of reference. In Chapters X and XIX we have found it necessary to make a convention as to the sense in which angles between arcs on the surface of the sphere are to be reckoned as positive. Such a convention is equally necessary in the present chapter; and we must add to it the further assumption that the triangle of reference is such that a point travelling from A to B, then from B to C, and then from C to A, along the sides, goes round the triangle in the conventionally positive sense.

As before, we select as positive the sense which appears counter-clockwise to an observer situated outside the sphere.

336. Convention with regard to small circles. The ambiguity that attaches to the conception of a small circle has been alluded to in Art. 201. An analytical treatment of the circle can only be rendered effective by making such a convention as will entirely remove this ambiguity.

A circle has two poles and, corresponding to them, two infinities of spherical radii represented by $\alpha + 2m\pi$, $\beta + 2n\pi$, where m and n may be any integers, and α, β represent the least arcs joining a point on the circle to the two poles. If we agree to exclude radii greater than π we have still two poles, and two radii α and β, supplements of one another.

To every circle, however, we shall assign arbitrarily a

certain direction or sense, in which it may be supposed to be described. That pole which would lie to the left of a person walking, on the outside of the sphere, along the circle in the assigned direction is defined to be *the* pole of the circle; the arc joining the pole to a point on the circle is defined to be *the* radius; and, of the two portions into which the circle divides the surface of the sphere, that which contains the pole is called the *inside* of the circle, whether it be greater or less than a hemisphere.

In other words, the spherical radius of a small circle may have any value between zero and π; but the circle is always to be regarded as having been described, (or swept out), by a radius turning about the pole in the counter-clockwise sense; and the direction in which the end of the radius moves is the direction belonging to the circle.

Thus, when a direction has been assigned to a circle, the pole is determined as lying to the left of that direction; and, conversely, if the pole be assigned, the direction is determined as corresponding to a counter-clockwise rotation round the pole.

The reader who has studied the theorems relating to coaxal and colunar systems of circles, discussed in Chapter X, will find it interesting to examine how far the scope of those theorems can be extended by the introduction of the convention of the present Article.

337. Definition of spherical tangent. By the spherical tangent at a point on a small circle is meant the great circle that has ordinary geometrical contact with the small circle, the direction assigned to the great circle being the same, at the point of contact, as that assigned to the small circle. In other words, if a great circle and a small circle touch one another, the pole of the small circle lies to the left of the great circle.

If the circles have geometrical contact, but their directions be such that the pole of the small circle lies to the right of the great circle, the circles meet at an angle π, and are not to be regarded as touching in the strict sense of the term.

In fact the only sort of contact contemplated in the definition is what may be called *internal* contact, the word "internal" being used in a sense corresponding to the definition, given in the preceding Article, of the "inside" of a circle.

338. Angle of intersection of two circles. If two circles A, B intersect in a point P, the angle between the tangents to the circles at P, measured in the positive sense of rotation from the tangent to A to the tangent to B, is called the angle of intersection of the circles at the point P. As it is necessary to choose one of the tangents as that from which the rotation begins, account must be taken of the order in which the circles are named. In general the circles, if they intersect at all, will do so in two points, say P and Q; and the angles of intersection at these two points together make up 2π.

Should it be desirable to distinguish between the two angles of intersection, we may do so by observing that at one of the points of intersection the circle A *enters* the circle B, and the circle B *emerges* from the circle A; at the other point the circle A emerges from the circle B, and the circle B enters the circle A. We shall agree to apply the term "the angle of intersection of two circles" to the angle of intersection at that point where the first circle emerges from the second.

When two circles touch, their angle of intersection is zero.

If A, B be the poles of two circles, and P a point of intersection, the angle APB is equal to the angle of intersection. Denoting this angle by ϕ, and the radii by r, s, we get, by applying the cosine formula to the triangle APB,

$$\sin r \sin s \cos \phi = \cos \mathrm{AB} - \cos r \cos s. \qquad \text{(3)}$$

339. Mutual Power of two circles. If A, B be the poles, and r, s the spherical radii of two circles, the expression

$$\cos \mathrm{AB} - \cos r \cos s. \quad \text{.....................(4)}$$

is called the *mutual power* of the circles.

Geometrical interpretation. When the circles intersect, formula (3) shews that their mutual power equals the product of the sines of their radii and the cosine of their angle of intersection.

When the circles do not intersect, their mutual power is not susceptible of this geometrical interpretation. In special cases other interpretations can be found; thus if the second circle be a great circle, the mutual power is $\cos \mathrm{AB}$, which equals $\sin p$, where p is the distance of the centre of A from the great circle, measured *inwards* along a spherical radius; if both circles be points, their mutual power is $-2\sin^2\frac{1}{2}\mathrm{AB}$; if the second circle be a point, and τ the length of the spherical tangent from it to A, $\cos \mathrm{AB} = \cos r \cos \tau$, and the mutual power is $-2\cos r \sin^2\frac{1}{2}\tau$.

The theorems which follow are, in several instances, expressed in terms of the angles of intersection of circles. They hold equally well when some of the pairs of circles do not intersect, or when some of the circles are points; but when this is the case it is necessary to express the theorems in terms of spherical powers instead of the non-existent angles of intersection, and there will be correspondingly different geometrical interpretations.

340. Let s_m, s_n be two circles, whose spherical radii are r_m, r_n, and whose poles, C_m, C_n, have for normal coordinates

$$(x_m, y_m, z_m) \text{ and } (x_n, y_n, z_n)$$

respectively. Then

$$\cos \mathrm{C}_m\mathrm{C}_n = x_m x_n + y_m y_n + z_m z_n,$$

and consequently if $\mathrm{P}(mn)$ represent the mutual power,

$$\mathrm{P}(mn) = x_m x_n + y_m y_n + z_m z_n - \cos r_m \cos r_n. \quad \text{..........(5)}$$

341. Theorem of the mutual powers. Let there be two systems, each consisting of five circles, namely, s_1, s_2, s_3, s_4, s_5 and $s'_1, s'_2, s'_3, s'_4, s'_5$; and let their radii, poles, etc, be represented by a notation corresponding to that of the previous Article.

Form the product of the two vanishing determinants

$$\begin{vmatrix} 0, & x_1, & y_1, & z_1, & \cos r_1 \\ 0, & x_2, & y_2, & z_2, & \cos r_2 \\ 0, & x_3, & y_3, & z_3, & \cos r_3 \\ 0, & x_4, & y_4, & z_4, & \cos r_4 \\ 0, & x_5, & y_5, & z_5, & \cos r_5 \end{vmatrix}, \quad \begin{vmatrix} 0, & x'_1, & y'_1, & z'_1, & -\cos r'_1 \\ 0, & x'_2, & y'_2, & z'_2, & -\cos r'_2 \\ 0, & x'_3, & y'_3, & z'_3, & -\cos r'_3 \\ 0, & x'_4, & y'_4, & z'_4, & -\cos r'_4 \\ 0, & x'_5, & y'_5, & z'_5, & -\cos r'_5 \end{vmatrix},$$

and we get immediately the following important result, first given by Prof. FROBENIUS:*

$$\begin{vmatrix} \mathrm{P}(11'), & \mathrm{P}(12'), & \mathrm{P}(13'), & \mathrm{P}(14'), & \mathrm{P}(15') \\ \mathrm{P}(21'), & \mathrm{P}(22'), & \mathrm{P}(23'), & \mathrm{P}(24'), & \mathrm{P}(25') \\ \mathrm{P}(31'), & \mathrm{P}(32'), & \mathrm{P}(33'), & \mathrm{P}(34'), & \mathrm{P}(35') \\ \mathrm{P}(41'), & \mathrm{P}(42'), & \mathrm{P}(43'), & \mathrm{P}(44'), & \mathrm{P}(45') \\ \mathrm{P}(51'), & \mathrm{P}(52'), & \mathrm{P}(53'), & \mathrm{P}(54'), & \mathrm{P}(55') \end{vmatrix} = 0 \quad \dots(6)$$

342. If each circle of one system intersect all the circles of the other system, the angle of intersection of s_m, s'_n being represented by (mn'), we derive the following theorem by substituting $\sin r_m \sin r'_n \cos(mn')$ for $\mathrm{P}(mn')$:

$$\begin{vmatrix} \cos(11'), & \cos(12'), & \cos(13'), & \cos(14'), & \cos(15') \\ \cos(21'), & \cos(22'), & \cos(23'), & \cos(24'), & \cos(25') \\ \cos(31'), & \cos(32'), & \cos(33'), & \cos(34'), & \cos(35') \\ \cos(41'), & \cos(42'), & \cos(43'), & \cos(44'), & \cos(45') \\ \cos(51'), & \cos(52'), & \cos(53'), & \cos(54'), & \cos(55') \end{vmatrix} = 0 \quad \dots(7)$$

343. Cases of orthogonal section If the first four circles of the first system are cut orthogonally by the same circle σ, and if we take σ to be the fifth circle of the second system,

* G. FROBENIUS, "*Anwendungen der Determinantentheorie auf die Geometrie des Maasses,*" *Crelle's Journal*, LXXIX, 1875, p. 187. See, however, CAYLEY, *Camb. Math. Journal*, II, 1841, p. 267, or *Collected Works*, Vol. I, p. 1.

the first four elements of the last column vanish in formula (7). Hence we get the following relation between the angles in which a system of four circles, possessed of a common orthogonal circle, are cut by any other four circles:

$$\begin{vmatrix} \cos(11'), & \cos(12'), & \cos(13'), & \cos(14') \\ \cos(21'), & \cos(22'), & \cos(23'), & \cos(24') \\ \cos(31'), & \cos(32'), & \cos(33'), & \cos(34') \\ \cos(41'), & \cos(42'), & \cos(43'), & \cos(44') \end{vmatrix} = 0 \quad \ldots\ldots(8)$$

If σ' be the circle which cuts s'_1, s'_2, s'_3 orthogonally, and if σ cut σ' orthogonally, σ' may be taken as the fourth circle of the first system. Then the first three elements of the last row vanish in formula (8), and we get the following relation between the angles of intersection of two systems of three circles, whose respective orthogonal circles cut at right angles:

$$\begin{vmatrix} \cos(11'), & \cos(12'), & \cos(13') \\ \cos(21'), & \cos(22'), & \cos(23') \\ \cos(31'), & \cos(32'), & \cos(33') \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots(9)$$

The condition that σ and σ' cut orthogonally is satisfied in the following cases:

(1) If the circles of the second system are coaxal; for, of the infinite number of circles which cut all the circles of a coaxal system orthogonally, there is always one which is orthogonal to another given circle such as σ.

(2) If the poles of the three circles of the second system lie on a great circle which passes through the pole of σ; for σ' is then this great circle.

(3) If σ and the three circles of the second system pass through a common point; for σ' is then the common point.

(4) If the six circles of the two systems pass through a common point; for σ and σ' are both point circles having the common point for pole, and $\mathsf{P}(\sigma\sigma')$ vanishes.

344. Condition that three circles, whose poles are on a great circle, be coaxal. In case (1) of the previous Article, since the circles orthogonal to a coaxal system contain an infinite number of pairs orthogonal to one another, it is legitimate to make the second system of three circles coincide with the first. We thus get as the condition that three circles, whose poles are on a great circle, be coaxal

$$\begin{vmatrix} \sin^2 r_1, & \mathrm{P}(12), & \mathrm{P}(13) \\ \mathrm{P}(21), & \sin^2 r_2, & \mathrm{P}(23) \\ \mathrm{P}(31), & \mathrm{P}(32), & \sin^2 r_3 \end{vmatrix} = 0 \text{ ....(10)}$$

When the circles cut, the mutual powers may be replaced by cosines, and each of the leading elements by unity. The resulting determinant has then factors of the form

$$\sin \tfrac{1}{2}\{(23) \pm (31) \pm (12)\}\,; \text{ ....(11)}$$

thus

$$(23) \pm (31) \pm (12) = 2n\pi, \text{ ....(12)}$$

where n is zero or an integer. The geometrical interpretation will be appreciated on examination of different diagrams of three circles passing through common points.

345. Circle cutting three coaxal circles. As a particular case of result (9), we may suppose the three circles of the first system to be coaxal, and the first and second circles of the second system to coincide respectively with the first and second of the first system. We then get a relation between the angles ϕ_1, ϕ_2, ϕ_3, at which any circle s is cut by three coaxal circles, s_1, s_2, s_3, namely,

$$\begin{vmatrix} 1, & \cos(12), & \cos\phi_1 \\ \cos(21), & 1, & \cos\phi_2 \\ \cos(31), & \cos(32), & \cos\phi_3 \end{vmatrix} = 0 \text{ ....(13)}$$

In this determinant the coefficient of $\cos\phi_1$ is

$$\cos(32)\cos(21) - \cos(31),$$

which is equal to $\sin(32)\sin(21)$, since $(31) = (32) + (21)$.

Removing the factor sin(12) from the complete expansion, we reduce the relation to the form

$$\sin(23)\cos\phi_1 + \sin(31)\cos\phi_2 + \sin(12)\cos\phi_3 = 0 \quad\ldots\ldots(14)$$

346. If the circle s of the previous Article be a point, we use, instead of (13), the corresponding determinant with mutual powers. Remembering that the mutual power of a point and a circle is $-2\cos r\sin^2\frac{1}{2}\tau$, when τ is the length of the tangent from the point to the circle, we get a relation between the tangents from any point to three coaxal circles, namely,

$$\begin{vmatrix} \sin^2 r_1, & \mathrm{P}(12), & \cos r_1 \sin^2\frac{1}{2}\tau_1 \\ \mathrm{P}(21), & \sin^2 r_2, & \cos r_2 \sin^2\frac{1}{2}\tau_2 \\ \mathrm{P}(31), & \mathrm{P}(32), & \cos r_3 \sin^2\frac{1}{2}\tau_3 \end{vmatrix} = 0 \quad\ldots\ldots\ldots(15)$$

If the point be on s_3, so that $\tau_3 = 0$, this relation expresses the fact that the sines of the halves of the tangents from a variable point on a circle to two other circles coaxal with it are in a constant ratio. (See Art. 177.)

347. Relations between five circles. In the theorem of the mutual powers let the two sets of circles coincide. Then it appears that any five circles satisfy the relation,

$$\begin{vmatrix} \sin^2 r_1, & \mathrm{P}(12), & \mathrm{P}(13), & \mathrm{P}(14), & \mathrm{P}(15) \\ \mathrm{P}(21), & \sin^2 r_2, & \mathrm{P}(23), & \mathrm{P}(24), & \mathrm{P}(25) \\ \mathrm{P}(31), & \mathrm{P}(32), & \sin^2 r_3, & \mathrm{P}(34), & \mathrm{P}(35) \\ \mathrm{P}(41), & \mathrm{P}(42), & \mathrm{P}(43), & \sin^2 r_4, & \mathrm{P}(45) \\ \mathrm{P}(51), & \mathrm{P}(52), & \mathrm{P}(53), & \mathrm{P}(54), & \sin^2 r_5 \end{vmatrix} = 0 \quad\ldots(16)$$

Thus the condition that four circles be cut orthogonally by a fifth is

$$\begin{vmatrix} \sin^2 r_1, & \mathrm{P}(12), & \mathrm{P}(13), & \mathrm{P}(14) \\ \mathrm{P}(21), & \sin^2 r_2, & \mathrm{P}(23), & \mathrm{P}(24) \\ \mathrm{P}(31), & \mathrm{P}(32), & \sin^2 r_3, & \mathrm{P}(34) \\ \mathrm{P}(41), & \mathrm{P}(42), & \mathrm{P}(43), & \sin^2 r_4 \end{vmatrix} = 0; \quad\ldots\ldots(17)$$

and the condition that four circles touch a fifth is (when they intersect one another),

$$\begin{vmatrix} 0, & \sin^2\tfrac{1}{2}(12), & \sin^2\tfrac{1}{2}(13), & \sin^2\tfrac{1}{2}(14) \\ \sin^2\tfrac{1}{2}(21), & 0, & \sin^2\tfrac{1}{2}(23), & \sin^2\tfrac{1}{2}(24) \\ \sin^2\tfrac{1}{2}(31), & \sin^2\tfrac{1}{2}(32), & 0, & \sin^2\tfrac{1}{2}(34) \\ \sin^2\tfrac{1}{2}(41), & \sin^2\tfrac{1}{2}(42), & \sin^2\tfrac{1}{2}(43), & 0 \end{vmatrix} = 0 \quad \text{..(18)}$$

348. Reciprocal Power of two circles The *mutual power of the reciprocals* of the circles s_m, s_n is clearly

$$\cos C_m C_n - \sin r_m \sin r_n \text{..................(19)}$$

or

$$x_m x_n + y_m y_n + z_m z_n - \sin r_m \sin r_n \text{..............(20)}$$

We shall call this the *reciprocal power* of the circles s_m, s_n, and denote it by $R(mn)$.

If the circles have a spherical common tangent of length $[mn]$, we find, on application of the cosine formula to the triangle whose vertices are the poles of the circles and of the tangent, that

$$R(mn) = \cos r_m \cos r_n \cos [mn]. \text{..............(21)}$$

349. Theorem of the reciprocal powers. Take two sets of five circles, and multiply together the vanishing determinants

$$(0, x_2, y_3, z_4, \sin r_5), \quad (0, x'_2, y'_3, z'_4, -\sin r'_5).$$

There results the following theorem, analogous to that of FROBENIUS:

$$\begin{vmatrix} R(11'), & R(12'), & R(13'), & R(14'), & R(15') \\ R(21'), & R(22'), & R(23'), & R(24'), & R(25') \\ R(31'), & R(32'), & R(33'), & R(34'), & R(35') \\ R(41'), & R(42'), & R(43'), & R(44'), & R(45') \\ R(51'), & R(52'), & R(53'), & R(54'), & R(55') \end{vmatrix} = 0. \quad (22)$$

350. The particular cases of this theorem correspond exactly to the particular cases of FROBENIUS'S theorem, which have been discussed in the preceding Articles; they are arrived at by analogous processes, and the forms in which they are stated

are deduced by substituting R for P. The geometrical interpretations, however, are different; for the angles of intersection in the one set of theorems are represented by lengths of common tangents in the other.

The theorems derived by *reciprocal powers* are, in fact, the reciprocals of the theorems derived by *mutual powers.* For example, if we substitute R for P throughout formula (10) of Art. 344, and cosines for sines in the leading elements, we get the condition that three circles whose poles are on a great circle should be colunar; and, corresponding to the theorems of Arts. 345, 346, we get properties of colunar circles.

Again the condition of Art. 347, that four circles should be touched by a fifth, would, if derived from the reciprocal power theorem, appear as an expression precisely like that in formula (18) in the squared sines of the halves of the common tangents. Dr. CASEY* points out that the condition is equivalent to

$$\sin\tfrac{1}{2}[23]\sin\tfrac{1}{2}[14]\pm\sin\tfrac{1}{2}[31]\sin\tfrac{1}{2}[24]\pm\sin\tfrac{1}{2}[12]\sin\tfrac{1}{2}[34]=0.\qquad(23)$$

From this result HART'S theorem may be deduced.

351. Four points and four circles. If s_1, s_2, s_3, s_4 be point circles, the sines of their radii vanish; consequently the determinant $(x_1,\ y_2,\ z_3,\ \sin r_4)$ vanishes. Multiplying it by the determinant $(x'_1,\ y'_2,\ z'_3,\ -\sin r'_4)$, we get the following relation between four points and four circles, wherein the reciprocal powers are replaced by cosines of tangents drawn from the points to the circles:

$$\begin{vmatrix} \cos[11'], & \cos[12'], & \cos[13'], & \cos[14'] \\ \cos[21'], & \cos[22'], & \cos[23'], & \cos[24'] \\ \cos[31'], & \cos[32'], & \cos[33'], & \cos[34'] \\ \cos[41'], & \cos[42'], & \cos[43'], & \cos[44'] \end{vmatrix}=0\ \ldots(24)$$

* *Proceedings of the Royal Irish Academy*, IX, 1866, p. 396. Cp. DARBOUX, *Annales de l'École Normale*, 2nd series, Vol. II, 1872, p. 347, and FROBENIUS, *loc. cit.*, p. 207.

352. Relation between the arcs joining four points. In this result let s'_1, s'_2, s'_3, s'_4 coincide with the points s_1, s_2, s_3, s_4. The formula then becomes a relation between the arcs joining any four points on the sphere. If three of the points be the corners of a triangle ABC, and if the arcs joining them to the fourth point be α, β, γ, the relation is

$$\begin{vmatrix} 1, & \cos c, & \cos b, & \cos \alpha \\ \cos c, & 1, & \cos a, & \cos \beta \\ \cos b, & \cos a, & 1, & \cos \gamma \\ \cos \alpha, & \cos \beta, & \cos \gamma, & 1 \end{vmatrix} = 0 \quad \text{.........(25)}$$

or $\Sigma \sin^2 a \cos^2 \alpha + 2\Sigma(\cos b \cos c - \cos a)\cos \beta \cos \gamma - 4n^2 = 0.$ (26)

When A, B, C are on a great circle, the sine of the triangle ABC vanishes; also $\sin(a \pm b \pm c) = 0$, and the left hand side becomes the square of $\Sigma \sin a \cos \alpha$. Thus we obtain the theorem of Art. 145.

353. The theorem of Art. 165 is also a particular case of relation (24). To obtain it we make s_1 and s'_1 coincide with the point A, s_2 and s'_2 with B, s_3 and s'_3 with C, s_4 with O, and s'_4 with P.

354. The present Chapter, based to a considerable extent on the memoir quoted in the foot-note to Art. 341, contains only a few of the more elementary of the numerous theorems which are obtained by the method of FROBENIUS. The reader who is interested in the subject should consult the original memoir, also Dr. CASEY'S paper referred to in Art. 350, and a paper by Dr. R. LACHLAN in the *Philosophical Transactions of the Royal Society*, Vol. 177, 1886.

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